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[Marco Caroccia]
CURRICULUM VITAE

INFORMAZIONI PERSONALI (NON INSERIRE INDIRIZZO PRIVATO E TELEFONO FISSO O CELLULARE)

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Curriculum Vitae et Studiorum

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Name	Marco
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Current employment

Dec. 2018 – Now **Post Doc**, *Research fellowship funded by Scuola Normale Superiore di Pisa and University of Florence.*

Previous employment

Sept. 2017 – Sept. 2018 **Post Doc**, *Universidade De Lisboa, Faculdade de Ciências - Centro de Matemática e Aplicações Fundamentais*, Campo Grande, Edifício C6, Piso 2, 1749-016 - Lisboa.

Sept. 2015 – Sept. 2017 **Post Doc**, *Carnegie Mellon University, Center for Nonlinear Analysis*, 5000 Forbes Avenue, Pittsburgh (PA), 15213 - USA.

Education

2011-2015 **Ph.D.**, *Università degli studi di Pisa*, Pisa, I received my Ph.D after the dissertation in July 27, 2015.
Ph.D. in Analysis

Thesis *On the isoperimetric properties of Planar N-clusters*

Supervisor Prof. Giovanni Alberti and Prof. Francesco Maggi

Summary We have studied some questions involving the N-clusters in $\mathbb{R}^n$. An N-cluster is a family of N-disjoint sets of finite perimeter.

2009–2011 **Master**, *Università degli studi di Firenze*, Firenze, *110/110.*
Applied Mathematics

Final work *Asymptotic inequalities for minimal planar clusters*

Supervisor Prof. Francesco Maggi

Summary Proof of the *Honeycomb Hexagonal Conjecture* and description of its consequence in the Cluster theory.

2006–209 **Bachelor**, *Università degli studi di Firenze*, Firenze, *110/110.*
General Curriculum, pure Mathematics

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Final work *Block Cipher Algorithm and primitive group*

Supervisor Prof. Orazio Puglisi

Summary Discussion of some algebraic results involving the security of the DES-AES cipher algorithms: Data Encryption Standard and Advanced Encryption Standard (US federal standard cipher algorithms adopted by the *National Security Agency* in 2009). Explanation of a “differential crypt-analysis algorithm ” testing the security of the DES code. Some consideration about the security of the AES code.

2001–2006 **General certificate of Secondary Education**, *Technical Institute -A. Meucci-* , Firenze, *100/100.*
Electronics and telecommunications industry expert foreman

List of publications

2019 *On the integral representation of variational functionals on BD.*
Caroccia - Focardi - Van Goethem, Preprint http://cvgmt.sns.it/paper/4404/ (Submitted to SIMA).

2019 *Mumford – Shah functionals on graphs and their asymptotics.*
Caroccia - Chambolle - Slepčev, Preprint arXiv:1906.09521 (Submitted to NONLINEARITY).

2018 *Damage-driven fracture with low-order potentials: asymptotic behavior, existence and applications*
Caroccia - Van Goethem, ESAIM: M2AN, 53 4 (2019) 1305-1350 doi: https://doi.org/10.1051/m2an/2019024

2017 *Equilibria configurations for epitaxial crystal growth with adatoms*
Caroccia - Cristoferi - Dietrich, Archive for Rational Mechanical Analysis (2018). https://doi.org/10.1007/s00205-018-1258-9

2017 *The Cheeger N-problem in terms of BV functions*
Caroccia - Littig, Journal of Convex Analysis, Volume 26, Num- ber 1 (2019).

2016 *On the isoperimetric properties of Planar N-clusters.*
Caroccia - Preprint arXiv:1601.07116

2015 *Cheeger N-clusters*
Caroccia - Calculus of variation and Partial Differential Equation (2017) 56: 30 doi:10.1007/s00526-017-1109-9

2014 *A sharp quantitative version of Hales' isoperimetric honeycomb theorem*
Caroccia, Maggi - Journal de Mathematiques Pures et Appliquees, 106.5 (2016): 935-956

2014 *A note on the stability of the Cheeger constant of N-gons.*
Caroccia, Neumayer - Journal of Convex Analysis, Volume 22, Number 4, pgg. 1207-1213 (2015)

Seminars

June 13, 2019 **Seminar**, *Universidade de Lisboa*, Center of Mathematics, Fundamental Applications and Operations Research - CmafCIO, Lisbon, Portugal.

Febr 06, 2019 **Conference Short talk**, *Levico Terme*, Bellavista Relax Hotel, Viale Vittorio Emanuele III 7, 38056 (Tn), Trento, Italy.

October 24, 2018	**Seminar**, *University of Jyväskylä*, Department of Mathematics and Statistics, Jyväskylä, Finland.
June 15, 2018	**Seminar**, *Università degli studi di Firenze*, Department of Mathematics and Statistics, Florence, Italy.
May 20-25, 2018	**Conference speaker**, *Banff international research station, Alberta, Canada*, Conference title: "Topics in the Calculus of Variations: Recent Advances and New Trends".
Apr 03, 2017	**Seminar**, *University of Jyväskylä*, Department of Mathematics and Statistics, Jyväskylä, Finland.
Apr 15, 2016	**Seminar**, *University of Leiden*, Lorentz Center, Leiden, Netherlands.
Jan 8, 2016	**Seminar**, *University of Cologne*, Mathematical Institute, Köln, Germany.
Oct 27, 2015	**Seminar**, *Carnegie Mellon University*, Center for Non Linear Analysis, Pittsburgh (PA), USA.
July 17, 2015	**Seminar**, *Max Planck Institute*, Max-Planck-Institute for mathematics in the sciences, Leipzig, Germany.
May 20, 2015	**Seminar**, *Università degli studi di Modena*, Dipartimento di Scienze Fisiche, Informatiche, Matematiche, Modena, Italy.

Research experiences

XXIX National conference in Calculus of Variation

Feb 04, 2019 – Feb 08, 2019: **Conference**, *Levico Terme*, Bellavista Relax Hotel, Viale Vittorio Emanuele III 7, 38056 (Tn), Trento, Italy.

Conference on calculus of variation organized by CIRM. I have had also the chance to give a short talk.

Emergence of Structures in Particle Systems: Mechanics, Analysis and Computation

Oct 28, 2018 – Nov 2, 2018: **Workshop**, *Oberwolfach, (GE)*, MFO, Oberwolfach Research Institute for Mathematics.

Topics in the Calculus of Variations: Recent Advances and New Trends

May 20, 2018 – May 25, 2018: **Workshop**, *Alberta(Canada)*, Banff international research station, Alberta, Canada, Workshop on Calculus of variation held at the international research station in Banff. I have had also the honor to be a speaker.

Topics in Applied Analysis and Optimisation

Dec 6, 2017 – Dec 8, 2017: **Conference**, *Lisbon (PO)*, Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, The event presented and discussed the main scientific interests among the research groups of the Weierstrass Institute in Berlin and mathematics centres in Portugal.

A Mathematical Tribute to Ennio De Giorgi

Sep 19, 2016 – Sep 23, 2016: **Conference**, *Pisa (IT)*, Centro di ricerca Ennio De Giorgi,Palazzo dei congressi, Via Giacomo Matteotti 1, Pisa(Pi), On the occasion of the vigintennial of Ennio De Giorgi's departure, the meeting gathers many generations of mathematicians and aims at the presentation of the most recent advances in the many research fields marked by De Giorgi's contributions.

New Frontiers in Nonlinear Analysis for Materials

Jun 2, 2016 Jun 10, 2016	**Summer school**, *Pittsburgh (PA)*, Center for Nonlinear Analysis, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh (PA), USA. PIRE-CNA 2016 Summer school
	New challenge for the calculus of variation stemming for problems in the materials science and image processing
May 16, 2016 May 20, 2016	**Workshop**, *Montréal, Quebec (CA))*, Centre de recherches mathématiques (CRM), Université de Montréal, PO Box 6128, Station Centre-ville, Montréal. Workshop in honor of the 60th birthday of Irene Fonseca
	Micro structure Evolution in Materials: Defects, Cracks & Interfaces
Apr 11, 2016 Apr 15, 2016	**Workshop**, *Leiden (NE)*, University of Leiden, Lorentz center, 3rd Floor Oort Building, Niels Bohrweg, Leiden (NE). Workshop on materials science.
	Analysis of partial differential equations
Dec 07, 2015 Dec 10, 2015	**Conference**, *Phoenix (AR)*, Double tree resort Hilton, Paradise Valley, Scottsdale, Phoenix (AR), US. SIAM Conference.
	Mathematics and Mechanics in the 22nd Century: Seven Decades and Counting
Oct 23, 2016 Oct 25, 2016	**Workshop**, *Eugene (OR)*, Valley River Inn Hotel, Eugene (OR), US. IMA Special Workshop in honor of the 90th birthday of Jerry Ericksen.)
	XXV National conference in Calculus of Variation
Feb 02, 2015 Feb 06, 2015	**Conference**, *Levico Terme*, Bellavista Relax Hotel, Viale Vittorio Emanuele III 7, 38056 (Tn), Trento, Italy. Conference on calculus of variation organized by CIRM.
	Fall Semester in Austin
Aug 07, 2014 Dec 18, 2014	**Courses and research**, *UT Texas at Austin*, 2515 Speedway, 78712 Texas, (512) 471-7711, Austin, Texas (US). I spent five months in Austin to work with my Ph.D thesis co-advisor Francesco Maggi. We worked on some stability isoperimetric issues involving the hexagonal tiling in the plane. I also attended the course: Geometric Measure Theory (Teacher Francesco Maggi).
	XXIV National conference in Calculus of Variation
Jan 26, 2014 Jan 31, 2014	**Conference**, *Levico Terme*, Bellavista Relax Hotel, Viale Vittorio Emanuele III 7, 38056 (Tn), Trento, Italy. Conference on calculus of variation organized by CIRM.
	School and workshop on "geometric measure theory and optimal transport"
Jul 15, 2014 Aug 03, 2014	**School and workshop**, *ICTP*, Strada costiera 11, 3415 (Ts), Trieste, Italy. The main topics of the school were the optimal transport theory and the theory of currents. The mini course on "optimal transport theory" were taught by Alessio Figalli and Guido De Philippis. As far as the course "theory of currents" is concerned the teachers were Camillo De Lellis and Emanuele Spadaro. At the end of the school I attend a one-week conference on these fields.
	Convexité cachée en équations aux derivées partielles non-linéaires
May 14, 2013 May 16, 2013	**Lectures series**, *Université catholique de Louvain*, Institut de recherche en mathématique et physique, Louvain La Neuve, Belgium. Course on optimal transport.

XXIII National conference in Calculus of Variation

Feb 3, 2013 – Feb 8, 2013 **Conference**, *Levico Terme*, Bellavista Relax Hotel, Viale Vittorio Emanuele III 7, 38056 (Tn), Trento, Italy.
Conference on calculus of variation organized by CIRM.

International Conference on Fluids And Variational Methods

Jan 28, 2013 – Feb 1, 2013 **Conference**, *University of Leipzig*, Felix-Klein-Hörsaal, Room 501, Paulinum, University of Leipzig, Universitätsstraße 1, 04109, Leipzig, Germany.
Conference on variational methods in fluids dynamics.

XXII National conference in Calculus of variation

Feb 5, 2012 – Feb 10, 2012 **Conference**, *Levico Terme*, Bellavista Relax Hotel, Viale Vittorio Emanuele III 7, 38056 (Tn), Trento, Italy.
Conference on calculcus of variation organized by CIRM.

Summer School

Jul 31, 2011 – Sept 2, 2011 **School**, *Universitá degli studi di Perugia*, Via Vanvitelli 1, 06123, Perugia, Italy.
In August 2011 I attended the "SMI", (Scuola Matematica interuniversitaria) in Perugia. I attended two lectures in analysis named Partial Differential Equation (given by prof. Pagani, Carlo and Johnson, Russell) and Functional Analysis (given by prof. Milmann, Vitali and Eidelman, Yuli)

International Modeling Week

June 14, 2010 – June 22, 2010 **School and workshop**, *Universidad Complutense de Madrid*, Plaza de Ciencias, 3 Ciudad Universitaria 28040, Madrid, Spain.
In June 2010 I attended the "International Modeling Week" in Madrid offered by the Universidad Complutense de Madrid trough a scholarship. The work, named "Calibration of single-factor HJM models of interest rates", was about a mathematical-economic model. Supervisors: Prof. *Gerardo Oleaga (UCM)*, *Miguel Carillon Alvarez (Banco Santander)*.

Event organized

Topics in Nonlinear Analysis: Calculus of Variations and PDEs

Oct 10, 2018 – Oct 12, 2018 **Conference**, *Lisboa (PT)*, Faculdade de Ciências, Universidade de Lisboa, Conference on Calculus of Variation - `https://sites.google.com/view/cvpdelisboa/home`.

Teaching experience

Escola de Verão de Matemática (summer school in mathematics)

Jun 20, 2018 – Jun 22, 2018 **Departamento de Matematica, Faculdade de Ciencias**, *Universidade de Lisboa*, Lisbon, PT, Calculus of Variations: birth and rise of a new discipline.
Three hours class for master students
`https://ciencias.ulisboa.pt/pt/escola-de-verao-de-matematica`

Principle of real analysis II - Spring 2017

Jan 2017 – May 2017 **Carnegie Mellon University**, *Mellon College of Science*, Pittsburgh (PA), US, Principle of real Analysis II.

Principle of real analysis I - Fall 2016

Sep 2016 – Dec 2016 **Carnegie Mellon University**, *Mellon College of Science*, Pittsburgh (PA), US, Principle of real Analysis I.

Principle of real analysis II - Spring 2016

Jan 2016 May 2016	**Carnegie Mellon University**, *Mellon College of Science*, Pittsburgh (PA), US, Principle of real Analysis II.
	Principle of real analysis I - Fall 2015
Sep 2015 Dec 2015	**Carnegie Mellon University**, *Mellon College of Science*, Pittsburgh (PA), US, Principle of real Analysis I.
	Analisi 1 - Fall 2013, Teaching Assistent
Sep 2013 Dec 2013	**Università degli studi di Pisa**, *Dipartimento di ingegneria civile*, Pisa, Analisi I.

Languages

English - writing	(comparable to) C1 level
English - speaking	(comparable to) C1 level
French - writing	(comparable to) A1 level
French - speaking	(comparable to) A2 level

Programming languages

C++	Sufficient
Mat lab	Good
ASM	Sufficient

July 28, 2019

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MAIN CONTRIBUTIONS AND FURTHER EXTENSIONS

M. CAROCCIA

Abstract. In this attachment we give a brief description of the main contributions and possible further extensions.

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1. Integral representation of local energy functionals on BD (with Matteo Focardi and Nicolas Van Goethem) - Cvgmt preprint

It is a classical problem in Calculus of Variations to determine the lower semicontinuous envelope of energies defined on subspaces of the space of functions with Bounded Deformation $BD(\Omega)$ (namely those functions $u:\Omega\to\mathbb{R}^n$ whose symmetric gradient $e(u)=\frac{1}{2}(\nabla u+\nabla u^t)$ is defined as a vector valued Radon measure Eu) in order to find the limits of minimizing sequences lying in the larger space $BD(\Omega)$. It is a well known fact [ACDM97] that $BD(\Omega)$ maps are approximately differentiable $\mathcal{L}^n$-a.e. in Ω, the jumps set is $\mathcal{H}^{n-1}$-rectifiable, and Eu can be decomposed as

$$Eu=e(u)\mathcal{L}^n\llcorner\Omega+[u]\odot\nu_u\mathcal{H}^{n-1}\llcorner J_u+E^cu,$$

where $e(u)=\frac{\nabla u+\nabla u^t}{2}$, ∇u being the approximate gradient of u, $[u]=u^+-u^-$ denotes the jump of u over the jump set J_u, $u^\pm$ being the traces left by u on J_u, ν_u is a unitary Borel vector field normal to J_u (here, $a\odot b:=\frac{1}{2}(a\otimes b+b\otimes a)$, a, $b\in\mathbb{R}^n$, denotes the symmetrized tensor product), and E^cu is the Cantor part of Eu.

The typical energy considered for relaxation ([Rin11], [ARDPR17], [Rin18]) are defined, for suitable integrands f, as

$$F(u):=\int_\Omega f(x,e(u)(x))\,\mathrm{d}x$$

on $LD(\Omega)$ (the space of maps $u\in L^1(\Omega;\mathbb{R}^n)$ with $e(u)\in L^1(\Omega;\mathbb{R}^n)$ and usually are set to be $+\infty$ otherwise. The $L^1(\Omega;\mathbb{R}^n)$ sequential lower semicontinuous envelope of the functional F, that is the greatest functional less or equal than F which is sequentially L^1 lower semicontinuous, is given by

$$\mathrm{Rel}F(u):=\inf\left\{\liminf_{j\to+\infty}F(u_j):\ u_j\to u\text{ in }L^1(\Omega;\mathbb{R}^n)\right\},$$

provided some coercivity assumptions on the integrands are imposed. Several progress has been done in the recent years in computing the L^1 semicontinuos envelope of such energies, also due to a huge improvement in the comprehension of the structure of the Cantor part of the measure Eu ([DPR16],[DPR17]) providing the analog of Alberti's rank-one theorem in the BV setting.

Theorem 1.1 ([DPR16])**.** *Let $u \in BD(\Omega)$. Then, for $|E^c u|$-a.e. $x \in \Omega$*

$$\frac{\mathrm{d}Eu}{\mathrm{d}|Eu|}(x) = \frac{\eta(x) \odot \xi(x)}{|\eta(x) \odot \xi(x)|} \tag{1.1}$$

for some ξ, $\eta : \Omega \to \mathbb{S}^{n-1}$ Borel vector fields.

This result allows also to approach the problem of relaxation throughout the *Global method* [BFM98], namely a procedure that, starting from very general properties of an L^1-lower semicontinuous energy $\mathcal{F}$, enables to give its integral representation in terms of the measures $\mathcal{L}^n, \mathcal{H}^{n-1}, |E^c u|$. With Prof. Focardi and prof. Van Goethem we have addressed such a question for given energies $\mathcal{F} : BD(\Omega) \times \mathcal{O}_\infty(\Omega) \to \mathbb{R}$ (here $\mathcal{O}_\infty(\Omega)$ stands for a family of regular open subsets of Ω) satisfying

(H1) $\mathcal{F}(\cdot; A)$ is $L^1(A; \mathbb{R}^n)$ lower semicontinuous for all $A \in \mathcal{O}_\infty(\Omega)$;

(H2) There exists a constant $C > 0$ such that for every $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$,

$$\frac{1}{C}|Eu|(A) \le \mathcal{F}(u; A) \le C(\mathcal{L}^n(A) + |Eu|(A)); \tag{1.2}$$

(H3) $\mathcal{F}(u; \cdot)$ is the restriction to $\mathcal{O}_\infty(\Omega)$ of a Radon measure for every $u \in BD(\Omega)$.

For technical reason we required also the following continuity properties

(H4) There exists a modulus of continuity Ψ such that

$$|\mathcal{F}(\mathrm{v} + u(\cdot - x_0); x_0 + A) - \mathcal{F}(u; A)| \le \Psi(|x_0| + |\mathrm{v}|)(\mathcal{L}^n(A) + |Eu|(A)) \tag{1.3}$$

for all $(u, A, \mathrm{v}, x_0) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega) \times \mathbb{R}^n \times \Omega$, with $x_0 + A \subset \Omega$;

(H5) There exists a modulus of continuity Ψ such that

$$|\mathcal{F}(u + \mathbb{L}(\cdot - x_0); A) - \mathcal{F}(u; A)| \le \Psi(|\mathbb{L}|\mathrm{diam}(A))(\mathcal{L}^n(A) + |Eu|(A)) \tag{1.4}$$

for every $(u, A, \mathbb{L}) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega) \times \mathbb{M}^{n\times n}_{skew}$, and for all $x_0 \in A$.

We have shown that the following Theorem integral representation result is in force.

Theorem 1.2 (C., Focardi,Van Goethem - 2019)**.** *Let $\mathcal{F} : D(\Omega) \times \mathcal{O}_\infty(\Omega)$ be an energy satisfying (H1)-(H5). Then, for all $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$ it holds*

$$\begin{aligned}\mathcal{F}(u; A) = &\int_A f(x, u(x), e(u)(x))\,\mathrm{d}x + \int_{J_u \cap A} g(x, u^-(x), u^+(x), \nu_u(x))\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &+ \int_A f^\infty\Big(x, u(x), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\Big)\,\mathrm{d}|E^c u|(x),\end{aligned}$$

where f and g can be explicitely computed and where

$$f^\infty(x_0, \mathrm{v}, \mathbb{A}) := \limsup_{t \to +\infty} \frac{f(x_0, \mathrm{v}, t\mathbb{A}) - f(x_0, \mathrm{v}, 0)}{t}, \tag{1.5}$$

Let us Remark that a similar result in the context of SBD maps (namely such maps $u \in BD(\Omega)$ with $E^c u = 0$) has been achieved in [ET03] under a more rigid invariance hypothesis (H5) which kills the possible dependence from u of the energy. The extension to the full BD case can be reached throughout the additional information provided by Theorem 1.1 on the Cantor part $E^c u$ of the measure Eu combined with a suitable selection of the blow-up sequence around a point $x \in \mathrm{spt}\, E^c u$.

After having carefully chosen the integrand f_0, it is possible to show that an energy of the type

$$F_0(u; A) := \int_A f_0(x, u(x), e(u)(x))\,\mathrm{d}x$$

defined on $LD(\Omega)$ has the nice property that $\mathrm{Rel}F_0$ satisfies (H1)-(H5). With Theorem 1.2 at hand we can thus provide the following relaxation result.

Theorem 1.3 (C., Focardi, Van Goethem - 2019)**.** *For all $(u, A) \in BD(\Omega) \times \mathcal{O}_\infty(\Omega)$ it holds*

$$\begin{aligned}\mathrm{Rel}F_0(u, A) = &\int_A f(x, u(x), e(u)(x))\,\mathrm{d}x \\ &+ \int_{J_u \cap A} g(x, u^-(x), u^+(x), \nu_u(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) + \int_A f^\infty\Big(x, u(x), \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\Big)\,\mathrm{d}|E^c u|(x),\end{aligned}$$

where f and g can be explicitely computed in terms of f_0.

Notice that this result allows to treat energies which might possibly depends on u.

2. Mumford-Shah functionals on graphs and their asymptotics (with Dejan Slepčev and Antonin Chambolle - [CCS19])

In collaboration with Dejan Slepčev we have studied a Γ-convergence problem involving a non-local approximation of the Mumford-Shah functional on graph. For a given cloud of points $\{x_1, \dots, x_n\} \subset \Omega$ chosen according to some distribution $\rho \in L^1$ (namely according to the Radon measure $\mu = \rho\mathcal{L}^n$) and for a given function $u : \{x_1, \dots, x_n\} \to \mathbb{R}$. By following the works [Gob98] and [GM01], we define the non-local Mumford-Shah for the graph as

$$\mathcal{F}_{\varepsilon,n}(u) := \frac{1}{\varepsilon}\frac{1}{n^2}\sum_{i,j=1}^{n} \arctan\left(\frac{|u(x_i) - u(x_j)|^2}{|x_i - x_j|}\right)\eta_\varepsilon(x_i - x_j)$$

where η_ε is a suitable Kernel vanishing at any scale bigger than ε. In the work [Gob98] the author shows that the family of functionals

$$\mathcal{F}_\varepsilon(u) := \frac{1}{\varepsilon}\int_{\Omega\times\Omega} \arctan\left(\frac{|u(x) - u(y)|^2}{|x - y|}\right)\eta_\varepsilon(x - y)\,\mathrm{d}x\,\mathrm{d}y$$

Gamma converges to the local Mumford - Shah

$$\mathcal{F}(u) := \lambda\int_\Omega |\nabla u|^2\,\mathrm{d}x + \mu\mathcal{H}^{n-1}(Su)$$

as was conjectured by Ennio De Giorgi. Here λ, μ are suitable dimensional constant and $u \in BV(\Omega)$. By exploiting this property and by suitably define a topology that allows us to compare functions definite on different graphs (namely the TL^p topology introduced by Dejan Slepcev and Nicolás García Trillos in [GTS16] and based on the optimal transport theory) we are able to show that, along a sequence of ε_n that decays to zero fast enough with respect to $n \to +\infty$, $\mathcal{F}_{\varepsilon_n,n}$ Gamma converges to the local Mumford-Shah suitably weighted with the distribution ρ. A similar work has been already developed for the non local total variation in [GTS16].

3. Damage-driven fracture with low-order potentials: asymptotic behavior, existence and applications (with Nicolas Van Goethem) [CVG17]

Given a body Ω, the potential energy of a deformation $u \in SBD(\Omega)$ is given by

$$\mathcal{F}(u) := \int_\Omega \mathbb{A}e(u)\cdot e(u)\,\mathrm{d}x + \int_{J_u} k(x, u^+, u^-, \nu(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) \tag{3.1}$$

as discussed in [FM98]. Here k is an energy density on the fracture that depends on the material (the fracture, or crack, is modeled by J_u) and $\mathbb{A} : M^{n\times n}_{sym} \to M^{n\times n}_{sym}$ is a fourth order tensor defined on the symmetric matrices. In [FI14] it has been proved that the sequence of energies

$$\mathcal{F}_\varepsilon(u, v) := \begin{cases} \displaystyle\int_\Omega \left[v\mathbb{A}e(u)\cdot e(u) + \frac{1}{\varepsilon}(1 - v)^2 + \varepsilon|\nabla v|^2\right]\mathrm{d}x & \text{if } (u, v) \in X_\varepsilon \\ +\infty & \text{otherwise} \end{cases} \tag{3.2}$$

(here $X_\varepsilon := H^1(\Omega;\mathbb{R}^n)\times W^{1,\infty}(\Omega;[\varepsilon,1]) \subset BD(\Omega)\times W^{1,\infty}(\Omega;[0,1])$) is Γ-converging to an energy of the type (3.1), more precisely to

$$\mathcal{F}(u,v) := \begin{cases} \displaystyle\int_\Omega \mathbb{A}e(u)\cdot e(u)\,\mathrm{d}x + a\mathcal{H}^{n-1}(J_u) + b\int_{J_u}\sqrt{\mathbb{A}[u]\odot\nu\cdot[u]\odot\nu} & \text{if } u\in SBD(\Omega), \\ & \text{and } v=1 \\ +\infty & \text{otherwise} \end{cases} \tag{3.3}$$

where a,b are dimensional constant and $v\odot w := \frac{1}{2}(v\otimes w + w\otimes v)$. In a recent work from [XVGN17] it has been derived a model that takes into account the presence of an high pressure fluid pumped into the crack in order to model the evolution of what is called the *hydraulic fracture*. The model considers a damage variable v as in (3.2) taking values ≈ 0 on a region $\omega\subset\Omega$. On this damaged region the authors consider the presence of a pressure p (induced by an incompressible fluid that is pushing through the material) that drives the evolution of the system (in particular of the damage). The stored energy in the damaged region is seen to be proportional to $p\mathrm{div}(u)$. In this setting, we see that a natural way to express this model is given by

$$\mathcal{G}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega \left[v\mathbb{A}e(u)\cdot e(u) + \frac{1}{\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2\right]\mathrm{d}x & \\ \displaystyle\qquad - \int_\Omega p(x,u,e(u),v)\mathrm{div}(u)\zeta(v)\,\mathrm{d}x & \text{if } (u,v)\in X_\varepsilon \\ +\infty & \text{otherwise} \end{cases} \tag{3.4}$$

where ε represent the size of the damaged region. In particular, this turns out to be a modification of the energy (3.1). By considering the pressure p depending also on u and v we can give a more general description of the phenomena introduced in [XVGN17]. This kind of problem has gained interests in the past due to the several applications that hydraulic fracture processes can find (as in the field of gas extraction or excavation). In several context the problem of the control of the evolution of cracks and/or fractures is a serious issue and that is where the model derived by Xavier, VanGoethem and Novotny in [XVGN17] can be useful for applications. As we discussed in [CVG17], further models can be recovered by exploiting the nonlinear potential F, as a plastics-slip models or some sort of averaged Tresca yield model.

An interesting question at this point is: to what kind of fracture problem the above sequence of energies converges to? With Nicholas Van Goethem we addressed it in a very general case. In particular, in [CVG17] we are able to provide a Γ-convergence result on the space $SBD^2(\Omega)\cap L^\infty(\Omega;\mathbb{R}^n)$ for the energy

$$\mathcal{F}_\varepsilon(u,v) := \begin{cases} \displaystyle\int_\Omega \left[v\mathbb{A}e(u)\cdot e(u) + \frac{\psi(v)}{\varepsilon}\right]\mathrm{d}x & \\ \displaystyle\qquad + \int_\Omega F(x,e(u),v)\,\mathrm{d}x & \text{if } (u,v)\in X_\varepsilon \\ +\infty & \text{otherwise} \end{cases}, \tag{3.5}$$

where F is a suitable convex (even non linear) potential subject to a sublinear growth hypothesis and ψ is a non increasing function such that $\psi(1)=0$. Notice that the hydraulic fracture model is recovered as soon as $F(x,M,v) = p(x,M,v)\mathrm{tr}\,(M)$. The limiting energy can be identified as

$$\mathcal{F}(u,v) := \begin{cases} \Phi(u), & \text{if } u\in SBD^2(\Omega) \\ & \text{and } v=1\ \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

where

$$\begin{aligned}\Phi(u) := &\int_\Omega \mathbb{A}e(u)\cdot e(u)\,\mathrm{d}x + \int_\Omega F(x,u,1)\,\mathrm{d}x\\ &+ a\int_{J_u}\sqrt{\mathbb{A}([u](z)\odot\nu(z))\cdot([u](z)\odot\nu(z))}\,\mathrm{d}\mathcal{H}^{n-1}(z)\\ &+ b\mathcal{H}^{n-1}(J_u) + \int_{J_u} F_\infty(z,[u]\odot\nu)\,\mathrm{d}\mathcal{H}^{n-1}(z).\end{aligned}$$

Here we have set

$$a = 2\sqrt{\alpha\psi(0)}, \qquad\qquad b = 2\int_0^1 \psi(t)\,\mathrm{d}t$$

and

$$F_\infty(z,M) := \lim_{t\to+\infty}\frac{F(z,tM,0)-F(z,0,0)}{t} \qquad \text{for } z\in J_u \text{ and } M\in M^{n\times n}_{sym}.$$

In particular we were able to show the following theorem.

Theorem 3.1. *The following assertion holds true.*

a) *For any $(u_\varepsilon, v_\varepsilon)\in H^1(\Omega;\mathbb{R}^n)\times V_\varepsilon$ such that $u_\varepsilon\to u$, $v_\varepsilon\to v$ in L^1 we have*
$$\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon,v_\varepsilon)\geq\mathcal{F}(u,v);$$
b) *For any $u\in SBD^2(\Omega)\cap L^\infty(\Omega;\mathbb{R}^n)$ there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ decreasing to 0 and $(u_{\varepsilon_k},v_{\varepsilon_k})\in H^1(\Omega;\mathbb{R}^n)\times V_{\varepsilon_k}$ such that*
$$u_{\varepsilon_k}\to u, \quad v_{\varepsilon_k}\to 1 \quad \text{in } L^1, \text{ and } \quad \lim_{k\to+\infty}\mathcal{F}_{\varepsilon_k}(u_{\varepsilon_k},v_{\varepsilon_k}) = \mathcal{F}(u,1).$$

The tools developed by [Cha04], [Iur14], [CFI17], [CC17], [AT90], [FI14] [Fon89] and [Fon88]are the starting point of our analysis.

We also underline that the analysis carried out in [Cri18] allows us to remove the L^∞ bound in the lim sup inequality and extend our Theorem to the whole $SBD(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$.

4. Equilibria configurations for epitaxial crystal growth with adatoms (with Riccardo Cristoferi and Laurent Dietrich) [CCD18]

Under the guidance of Irene Fonseca and Giovanni Leoni and together with two CMU post-docs, Riccardo Cristoferi and Laurent Dietrich, we studied a time-dependent system of PDEs modeling the evolution of a solid-vapour interface under the presence of an adatom density. The system of equations has been derived firstly in [FG03] within the framework of configurational forces, and formally it read as

$$\begin{cases}\partial_t u &= \Delta_{\Sigma_t}\mu - V(uH+1), & \text{on } \Sigma_t\\ V &= -\psi(u)H + \mu(uH+1), & \text{on } \Sigma_t\end{cases} \tag{4.1}$$

where $\{\Sigma_t\}_{t\in I}$ is the family of interfaces representing the evolution of the material (typically the top part of the boundary of a domain Ω_t representing the bulk material as in Figure 4). Here $u(\cdot,t):\Sigma_t\to\mathbb{R}^+$ is the adatom density, Δ_{Σ_t} is the Laplace-Beltrami operator, $V(x,t)$ is the normal velocity to Σ_t at $x\in\Sigma_t$, $H(x,t)$ is the mean curvature of Σ_t at $x\in\Sigma_t$, ψ is the surface free energy, and $\mu=\psi'(u)$ is the chemical potential. This model arises as a refinement of the surface diffusion model

$$V = \Delta_{\Sigma_t}\mu,$$

with the introduction of an adatom density u on Σ_t and the mass constraint

$$|\Omega_t| + \int_{\Sigma_t} u(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \text{constant in time}$$

which do not exclude a priori that part of the material may become part of the adatoms and conversely. In the paper of Burger [B+06] the author shows how to variationally recover (as a gradient flow in a suitable topology) the model (4.1) starting from the energy

$$E(\Omega,u) := \int_{\partial\Omega}\psi(u)\,\mathrm{d}\mathcal{H}^{n-1}(x) \tag{4.2}$$

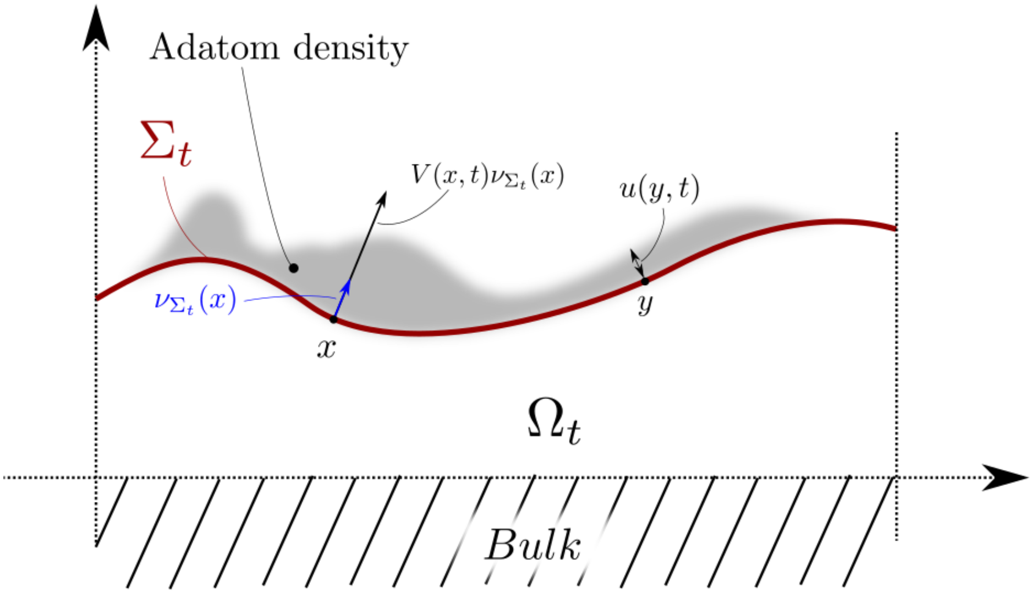

FIGURE 4.1. The situation modeled by (4.1): a deposition of material Ω_t on the bulk with a solid-vapor interface Σ_t and an adatoms concentration on the top. The system evolves by moving in the normal direction ν_{Σ_t} to Σ_t.

where Ω has finite volume, $\partial\Omega(=\Sigma$ the diffusing interface) has finite $\mathcal{H}^{n-1}$ measure and ψ is a convex, superlinear function such that $\psi(x) > 1$. In [RV06] the authors conjectured that the phase field approximation, inspired by the Cahn-Hiliard approach,

$$E_\varepsilon(\phi, u) := \int_{\mathbb{R}^n} \psi(u) \left(\frac{\varepsilon}{2} |\nabla\phi|^2 + \frac{1}{\varepsilon} W(\phi) \right) \mathrm{d}x \tag{4.3}$$

should yield a system of PDEs (depending on ε) whose solutions converges to the solutions of (4.1). In the above phase field approximation we ask that ϕ is a Sobolev function, u is a summable function and W is a suitable potential. The authors also show formal convergence of the equations. A serious issue in these kind of approximation is represented by the lower semi-continuity of the underlying energy $E(\Omega, u)$ and one of the problem that immediately arises in dealing with the energy (4.2) is that the natural topology, the one that gives compactness, does not guarantees any kind of lower semi-continuity. In order to solve this problem and obtain an hint for what concern the eventual limit of (4.3) we consider the energy (4.2) as an energy defined on the space $\mathcal{S}$ of the couple (Ω, μ) where Ω is a *set of finite perimeter* and μ is a Radon measure concentrated on $\partial^*\Omega$ (a set contained in $\partial\Omega$ and coinciding with $\partial\Omega$ whenever $\partial\Omega$ is smooth). In this context μ represents the adatom density. The natural topology in this space is the one induced by the L^1 convergence in the variable Ω and the weak star convergence in the variable μ. By endowing the space $\mathcal{S}$ with such a topology we are not only able to study critical points of E and the behavior of the time-independent version of (4.1), but also to characterize the lower semi-continuous envelope of E. Our main result concerning this characterization can be stated as follows.

Theorem 4.1. *Let $\mu = u\mathcal{H}^{n-1} \llcorner \partial^*\Omega + \mu^s$, for any couple (Ω, μ), be the Radon-Nikodym decomposition of μ. Then the functional*

$$F(\Omega, \mu) = \int_{\partial^*\Omega} \overline{\psi}(u)\, \mathrm{d}\mathcal{H}^{n-1}(x) + \Theta\mu^s(\mathbb{R}^n)$$

is the lower semi-continuous envelope of E.

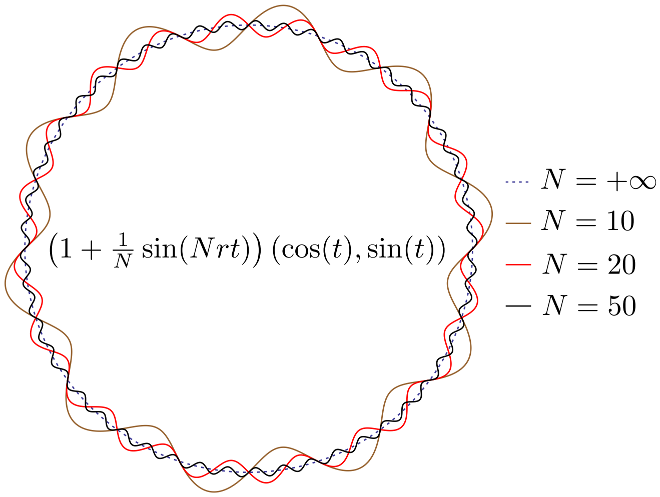

FIGURE 4.2. By oscillating at a faster and faster frequency and with smaller and smaller amplitude the perimeter increases and the sequence still converges in L^1 to Ω (in this case Ω stands for the unit ball or the plane). Due to the fact that $\psi(x) > 1$ this process allows us to compensate an eventual loss of adatom density u on $\partial\Omega$ by simply increasing the surface area. This concentration phenomena lead to a limiting couple that has energy $F < E$.

In the above theorem $\overline{\psi}$ stands for the convex subadditive envelope of ψ and

$$\Theta = \lim_{x\to+\infty} \frac{\bar{\psi}(x)}{x}.$$

Under the light of this new result we can infer now that the phase field approximation (4.3) might actually not be the correct one for the sharp model (4.2). Indeed classical results lead us to say that, if there is a Γ-limit (a particular notion of variational convergence) for the approximate energy (4.3), then such a limit needs to be lower semicontinuous. In particular the functional $F(\cdot,\cdot)$ is more likely to be the Γ-limit of E_ε rather than E. This open to the possibility that the solution v_ε of the gradient flows for the approximate energy E_ε could converge to the gradient flow for the energy F. The lack of semi-contuinuity is mostly due to concentration phenomena that take place on the boundary of Ω (see picture 4). By exploiting this kind of behavior it is possible to produce sequences $(\Omega_k,\mu_k) \to (\Omega, u\mathcal{H}^{n-1}\llcorner\partial^*\Omega)$ that approximates $F : E(\Omega_k,\mu_k) \to F(\Omega,u) < E(\Omega,u)$, called *recovery sequence*. It is not surprising, then, that the numerical simulation developed in [SV08] shows features that mimic the behavior of the recovery sequences in the relaxation process: the sequences follow the path that minimizes the energy.

Under the light of Theorem 4.1 we are now able to prove the the Γ-convergence of the energy E_ε (under the topology of the space $\mathcal{S}$) to the energy F). We expect that our future research directions will lead us towards the understanding of the behavior (in ε) of the solutions of the system of PDEs induced by E_ε and to the short-time existence of solution for the system 4.1. We would like to underline that, in this context, many techniques can be imported from the variational framework in order to discuss issues such as existence and regularity of solutions. For instance we can try to look for a gradient flow point of view as it has already been successfully implemented by Fonseca, Fusco, Leoni and Morini for the treatment of surface diffusion-type models in various frameworks (see [FFLM07],[FFLM12], [FFLM15]). Let us recall also that a work that contains several new ideas in this sense is [FJM17] where the authors provides short-time existence for a surface diffusion system of PDEs.

5. A sharp quantitative version of Hales' isoperimetric honeycomb theorem (joint work with Francesco Maggi - UT Texas at Austin) [CM16]

In 2001 Thomas Hales solved the hexagonal honeycomb conjecture, and established that among all the partition of the plane in unit-area chambers the hexagonal tiling provides the partition with minimum local perimeter ([Hal01]). This results can be translated in an isoperimetric inequality on the flat thorus (where the problem can be stated nicely). In particular, given a suitable flat thorus $\mathcal{T}$ and any partition $\mathcal{E}$ of $\mathcal{T}$ in unit-area chambers, it holds that

$$P(\mathcal{E}) \geq P(\mathcal{H})$$

where $\mathcal{H}$ is the hexagonal honeycomb partition of $\mathcal{T}$. By using a selection principle for planar N-clusters, developed in [CLM14] following the scheme given in [CL12], we were able to prove the sharp stability inequality

$$P(\mathcal{E}) \geq P(\mathcal{H})(1 + \kappa\alpha(\mathcal{H})^2)$$

where $\alpha(\cdot)$ is a suitable L^1-type distance of $\mathcal{E}$ from $\mathcal{H}$ and k is a constant depending on $\mathcal{T}$. Our main result is the following theorem.

Theorem 5.1. *There exists a positive constant κ depending on $\mathcal{T}$ such that*

$$P(\mathcal{E}) \geq P(\mathcal{H})\left\{1 + \kappa\,\alpha(\mathcal{E})^2\right\}, \tag{5.1}$$

whenever $\mathcal{E}$ is a unit-area tiling of $\mathcal{T}$ and

$$\alpha(\mathcal{E}) = \inf \mathrm{d}(\hat{\mathcal{E}}, v + \mathcal{H})$$

where the minimization takes place among all $v = (t\sqrt{3}\ell, s\ell)$, $s,t \in [0,1]$, and among all tilings $\hat{\mathcal{E}}$ obtained by setting $\hat{\mathcal{E}}(h) = \mathcal{E}(\sigma(h))$ for a permutation σ of $\{1,...,N\}$.

6. Equidistribution of the energy for an isoperimetric partition problem with fixed boundary (with Giovanni Alberti - Università degli studi di Pisa)[Car16]

A conjecture due to Morgan and Heppes (see [HM05]) states that the global shape of perimeter minimizer planar N-clusters having equal-volume chambers, suitably normalized must converge, in the L^1-sense, to a ball. The global shape should be intend as $\mathcal{E}(0)^c$, where $\mathcal{E}(0)$ is the external chamber of the cluster $\mathcal{E}$. So far no progress has been made in proving this conjecture and the reason could lie in the difficulties arising when we try to understand in which sense the shape of the internal chambers has an influence on the global shape of perimeter minimizers N-clusters. In order to investigate the influence of the boundary on the internal structure of perimeter minimizers N-clusters, it makes sense to consider an isoperimetric problem on planar N-cluster with fixed boundary. Namely, for a fixed set Ω with finite perimeter we consider the quantity

$$\rho(N,\Omega) := \inf_{\mathcal{E}\in \mathrm{Cl}(N,\Omega)} \{P(\mathcal{E})\}, \tag{6.1}$$

where the infimum is taken among all the N*-clusters of* Ω, to be precise:

$$\mathrm{Cl}(N,\Omega) := \left\{\mathcal{E} \text{ planar } N\text{-cluster with } |\mathcal{E}(j)| = \frac{|\Omega|}{N} \text{ for } j \neq 0 \text{ and } \mathcal{E}(0) = \Omega^c\right\}. \tag{6.2}$$

In order to better understand the behavior of the localized energy $P(\mathcal{E}; Q_l)$, where $Q_l \subset\subset \Omega$ is a square of edge-length l and $\mathcal{E}$ is a minimizing N-cluster for an open set Ω, we provide an "equidistribution theorem" in the spirit of the one obtained by Alberti, Choksi e Otto in [ACO09]:

There exists a universal constant C such that for every open bounded set Ω, every $\mathcal{E} \in Cl(N,\Omega)$ minimizing N-cluster for Ω and every closed cube $Q \subset\subset \Omega$ "far enough" from the boundary and "large enough with respect to the size of the chambers" the following holds:

$$\Big|P(\mathcal{E};Q) - |Q|\frac{P(H)}{2}\sqrt{\frac{N}{|\Omega|}}\Big| \leq CP(Q). \tag{6.3}$$

where H denotes a unit-area regular hexagons.

From a qualitative point of view, estimate (6.3) gives us information about the average energy of $\mathcal{E}$ inside the cube Q. If we divide both members of (6.3) by $\frac{|Q|}{|\Omega|}N$, which represents the expected number of chambers of $\mathcal{E}$ lying inside Q, we obtain

$$\Big|\frac{P(\mathcal{E};Q)}{\frac{|Q|}{|\Omega|}N} - \frac{P(H)}{2}\sqrt{\frac{|\Omega|}{N}}\Big| \leq \frac{CP(Q)}{|Q|}\frac{|\Omega|}{N}.$$

Note that $\frac{P(H)}{2}\sqrt{\frac{|\Omega|}{N}}$ is the average energy of a uniform grid of hexagons H having volume $\frac{|\Omega|}{N}$ (and thus perimeter $P(H)\sqrt{\frac{|\Omega|}{N}}$). The exact statement is the following.

Theorem 6.1. *Let Ω be an open bounded set with Lipschitz boundary and $0 \le \beta < \frac{1}{2}$ be a positive real number. Then there exist three positive constant η, λ, C depending only on β and on the shape of Ω with the following property. For every cube $Q_l \subset\subset \Omega$ with*

$$d(\partial Q_l, \partial\Omega) > \eta\sqrt{|\Omega|}N^{-\frac{1}{6}}, \qquad l \ge \lambda\sqrt{|\Omega|}N^{-\beta} \tag{6.4}$$

and for every indecomposable minimizing N-cluster $\mathcal{E} \in Cl(N,\Omega)$ the following holds

$$\left|P(\mathcal{E};Q_l) - \frac{P(H)}{2}|Q_l|\sqrt{\frac{N}{|\Omega|}}\right| \le CP(Q_l)^{\frac{3}{2}}\left(\frac{N}{|\Omega|}\right)^{\frac{1}{4}}. \tag{6.5}$$

The previous statement refers to indecomposable minimizing N-clusters, which are those N-clusters whose chambers are indecomposable. We are trying to remove this assumption. We underline that under the stronger hypothesis of equi-boundedness of diameters of the chambers of a perimeter minimizer N-cluster we can prove the following stronger theorem.

Theorem 6.2. *Let Ω be an open and bounded set with finite perimeter. There exists a universal constant $C > 0$ with the following property. For every $\mu \ge \mathrm{diam}(H)$, every closed cube $Q_l \subset\subset \Omega$ such that*

$$d(\partial Q_l, \partial\Omega) > 4\mu\sqrt{\frac{|\Omega|}{N}}, \qquad l \ge 6\mu\sqrt{\frac{|\Omega|}{N}} \tag{6.6}$$

and every $\mathcal{E} \in Cl(N,\Omega)$ μ-bounded minimizing N-cluster the following holds:

$$\left|P(\mathcal{E};Q_l) - |Q_l|\frac{P(H)}{2}\sqrt{\frac{N}{|\Omega|}}\right| \le CP(Q_l)\mu. \tag{6.7}$$

Here a μ-bounded minimizing N-cluster is perimeter minimizer N-cluster such that

$$\max_{i=1,\dots,N}\{\mathrm{diam}(\mathcal{E}(i))\} \le \mu.$$

7. Cheeger N-clusters [Car17]

As a generalization of the classical Cheeger constant in my Ph.D. thesis we have considered the Cheeger constant for N-clusters

$$H_N(\Omega) = \inf\left\{\sum_{i=1}^{N}\frac{P(\mathcal{E}(i))}{|\mathcal{E}(i)|}\Big|\mathcal{E} \subset\subset \Omega \text{ is an } N\text{-cluster}\right\} \tag{7.1}$$

where Ω is an open bounded set. Any $\mathcal{E} \subset \Omega$ attaining the above infimum is said to be a Cheeger N-cluster of Ω. It can be shown that the existence of Cheeger N-cluster for any open bounded set is always guaranteed. The quantity H_N seems to represent the right object to study in order to provide some non trivial lower bound on the optimal partition functional

$$\Lambda_N^{(p)}(\Omega) := \inf\left\{\sum_{i=1}^{N}\lambda_1^{(p)}(\mathcal{E}(i))\right\}, \tag{7.2}$$

where $\lambda_1^{(p)}$ denotes the first Dirichlet eigenvalue of the p-Laplacian, defined as

$$\lambda_1^{(p)}(E) := \inf\left\{\int_E |\nabla u|^p \, dx \;\Big|\; u \in W_0^{1,p}(E),\ \|u\|_{L^p} = 1\right\}.$$

The infimum in (7.2) is taken over all the N-clusters $\mathcal{E}$ whose chambers are *quasi-open sets of* Ω. The importance of the partition problem (7.2) relies on the fact that it provides a way to look at the asymptotic behavior in N of the N-th Dirichlet eigenvalue of the classical Laplacian (the 2-Laplacian), as Caffarelli and Lin showed in [CL07]. In [CL07], Caffarelli and Lin prove that there exist two constants C_1 and C_2 depending only on the dimension such that

$$C_1\frac{\Lambda_N^{(2)}(\Omega)}{N} \le \lambda_N^{(2)}(\Omega) \le C_2\frac{\Lambda_N^{(2)}(\Omega)}{N}. \tag{7.3}$$

The detailed study of $\lambda^{(2)}_N(\Omega)$ for $N \geq 2$ seems to be a hard task (so far only the case $N = 1, 2$ are known, see for instance [Hen06]), and this is why the asymptotic approach suggested by Caffarelli and Lin could be a good way to look at the spectral problem. Caffarelli and Lin's conjecture (appearing in [CL07]) about the asymptotic behavior of $\Lambda^{(2)}_N(\Omega)$ in the planar case states that

$$\Lambda^{(2)}_N(\Omega) = \frac{N^2}{|\Omega|}\lambda^{(2)}_1(H) + o(N^2),$$

where H denotes a unit-area regular hexagon. So far no progress has been made in proving the conjecture, but numerical simulations (see [BBO09]) indicate that the conjecture could be true. We note here that the constant H_N is the analogous of the Cheeger constant in the optimal partition problem for p-laplacian eigenvalues. Indeed we can always give a lower bound on $\Lambda^{(p)}_N$ by making use of Cheeger's inequality and Jensen's inequality:

$$\Lambda^{(p)}_N(\Omega) \geq \frac{1}{N^{p-1}}\left(\frac{H_N(\Omega)}{p}\right)^p. \tag{7.4}$$

By combining (7.4) with a comparison argument, we are also able to compute the limit as p goes to 1 and obtain

$$\lim_{p\to 1} \Lambda^{(p)}_N(\Omega) = H_N(\Omega). \tag{7.5}$$

Thus, the constant H_N seems to provide the suitable generalization of the Cheeger constant for the study of $\Lambda^{(p)}_N$. We point out that the asymptotic behavior of H_N is

$$\frac{h(B)\sqrt{\pi}}{\sqrt{|\Omega|}} \leq \liminf_{N\to+\infty} \frac{H_N(\Omega)}{N^{\frac{3}{2}}} \leq \limsup_{N\to+\infty} \frac{H_N(\Omega)}{N^{\frac{3}{2}}} \leq \frac{h(H)}{\sqrt{|\Omega|}}. \tag{7.6}$$

In particular, the following is conjectured in [Car17]

$$\lim_{N\to+\infty} \frac{H_N(\Omega)}{N^{\frac{3}{2}}} = \frac{h(H)}{\sqrt{|\Omega|}}, \tag{7.7}$$

which is just Caffarelli and Lin's conjecture for the case $p = 1$. Let us underline that in [BFVV17] relation (7.7) has been proven as a consequence of a general theorem. In particular the authors of [BFVV17] built a strong link between the Pólya-Szegö (conjectured) inequality and the partition problem for the Laplacian eigenvalues achieving a huge improvement in the direction of proving the Caffarelli and Lin's conjecture.

In the paper [Car17] we mainly focused on the general structure and regularity of Cheeger N-clusters in order to lay down the basis for future investigations on H_N. The statements involving regularity are quite technical, and we just point out here that if $\mathcal{E}$ is a Cheeger N-cluster of Ω the following (heuristical) statement holds.

Theorem. *For every $i = 1, \ldots, N$ the reduced boundary of each chambers $\partial^*\mathcal{E}(i) \cap \Omega$ is a $C^{1,\alpha}-$hypersurfaces (for every $\alpha \in (0, 1)$) that is relatively open inside $\partial\mathcal{E}(i) \cap \Omega$ and has piecewise constant mean curvature. Furthermore it is possible to characterize the singular set of a Cheeger N-cluster $\mathcal{E}$ as a suitable collection of points with density zero for the external chamber*

$$\mathcal{E}(0) = \Omega \setminus \bigcup_{i=1}^N \mathcal{E}(i).$$

Moreover if the dimension is $n = 2$ the singular set is discrete and the chambers $\mathcal{E}(i) \subset\subset \Omega$ are indecomposable.

Figure 7 illustrates graphically the structures of the Cheeger N-clusters suggested by [**?**, Theorem 2.1, 2.3, 2.7].

Let me underline that an additional contribution in this field it has been given in collaboration with Samuel Littig in [CL17] where, following the approach suggested in [CL07], we propose a minimization problem equivalent to (7.1) which is stated for vector valued BV functions.

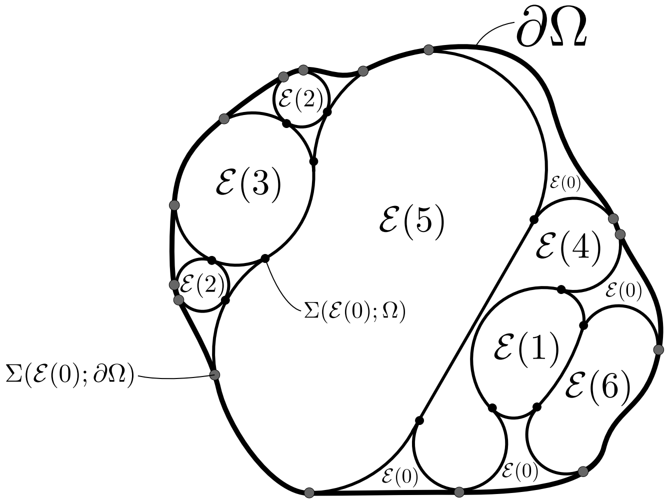

FIGURE 7.1. An example of a possible Cheeger 6-cluster in dimension $n = 2$ suggested by [Car17, Theorem 2.1, 2.3, 2.7]. The sets $\Sigma(\mathcal{E}(0);\partial\Omega), \Sigma(\mathcal{E}(0);\Omega)$ are the singular sets.

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MATERIALS SCIENCE, NON LOCAL INTERACTIONS AND MINIMALITY: BETWEEN PURE AND APPLIED MATH

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Abstract. My research interests range across different areas of mathematical analysis and calculus of variation, starting from geometric measure theory and partition problems, treated in my Ph.D thesis, and ending on my recent analysis of systems of nonlinear PDEs having a gradient flow structure. Common to all these project is that they can be treated by exploiting a variational approach and the direct method of calculus of variation.

I here propose two directions of investigation (explained in Section 1, 2) that requires all the formalism and the techniques coming from Calculus of Variation to be addressed. I do stress the fact that, even if the two paths of investigation are distinct and indeed treat different questions and issues, the underlying instruments are the same. The two paths are thought to be followed together contemporaneously. The idea is that having more than one direction of research is keeping active my interest in more than one discipline and can open the doors to possible further collaboration with a bigger network of scientists. I do also list in Section 3 some problems I am dedicating to, that are not embedded in a bigger framework but that can be achieved with a reasonable amount of effort in a short time, possibly opening to eventual generalization.

The first part, Section 1, concerning function of Bounded Deformation, is relative to the theory of fractures and its way of being modeled in the recent years. The second part, Section 2, instead treats the planar minimal clusters of the plane, that has several connection with models in materials science from crystallization to the theory of grain boundaries evolution.

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1. Function of bounded deformation: beyond fracture models

Given a body Ω and a deformation $u : \mathbb{R}^n \to \mathbb{R}^n$, the energy density of such a deformation, in the regime of linear elasticity, is given by $\mathbb{A}e(u) \cdot e(u)$, where $\mathbb{A} : \mathbb{R}^n \to M^{n\times n}_{sym}$ is a symmetric matrix-valued map depending on the elastic properties of the material and $e(u) = \frac{\nabla u + \nabla u^t}{2}$ is the symmetric part of the gradient of u (we refer to [Gou94] for a detailed justification of why we can assume such a behavior for the description of the deformation process). In particular, following the approach suggested in [FM98], it is possible to model the evolution and the equilibria configuration of a fractured body, starting from a deformation $u \in BV(\Omega;\mathbb{R}^n)$, by associating it to a potential energy of the type

$$\mathcal{F}(u) := \int_\Omega \mathbb{A}e(u)\cdot e(u)\,\mathrm{d}x + \int_{J_u} k(x,u^+,u^-,\nu(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) \tag{1.1}$$

where k is an energy density on the fracture that depends on the material as well (here the fracture, or crack, is modeled by J_u). These models describe very well the fracturing behavior, indeed if at a point x the external forces or the boundary conditions pushes the deformation to have a very high elastic energy, then the system will try to collapse into a crack in order to decrease $\mathcal{F}(u)$. The presence of just the symmetric part of the gradient in the energy leads us to adopt a space of deformation slightly bigger than BV. In particular the natural space where to define these kind of energies is the space of function of *bounded deformation* set to be, in analogy with the space of BV functions, as the family of all the L^1 function such that $e(u)$ is defined in the distributional sense. In particular, $u \in BD(\Omega)$ if $u \in L^1$ and if exists a vector-valued Radon measure Eu such that for all $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ with $\|\varphi\|_\infty \leq 1$ it holds

$$\sum_{j=1}^n \int_\Omega (e(\varphi))_{i,j}u_j\,\mathrm{d}x = \sum_{j=1}^n \int_\Omega \varphi_j(x)\,\mathrm{d}(Eu)_{i,j}(x), \qquad \text{for all } i=1,\dots,n$$

As shown in [ACDM97], given a function $u \in BD(\Omega)$ it is possible to decompose the measure Eu in three parts:

$$Eu = e(u)\mathcal{L}^n + [u]\odot\nu\mathcal{H}^{n-1}\llcorner J_u + E^c u$$

where $e(u)$ is the density of the absolutely continuous part of Eu, J_u is the jump set of u, ν is a unitary vector field normal to J_u (which is well defined since J_u is $(n-1)-$rectifiable), $[u]=u^+-u^-$ is defined $\mathcal{H}^{n-1}$-a.e. on J_u and $a\odot b := \frac{1}{2}(a\otimes b + b\otimes a)$. If $E^c u = 0$ then u is said to be a function of *special* bounded deformation $u \in SBD(\Omega)$. In this terms is clear that the energy defined in (1.1) is still well defined for all $u \in SBD(\Omega)$. Compactness and semicontinuity, under reasonable assumption on $\mathbb{A}$ and k, is guaranteed on this space thanks to the work of [BCDM98]. The ideas in this part of the research project deals with some issues concerning energies defined on $BD(\Omega), SBD(\Omega)$ and the structure of such spaces.

1.0.1. *State of art and main questions.* The importance of a good approximation for energies of the type (1.1) is crucial in terms of numerical simulations or in term of minimization processes. One of the most known and used tools in this sense is the concept of Γ-convergence. In [FI14] it has been proved that a good approximation for 1.1 is provided by

$$\mathcal{F}_\varepsilon(u,v) := \int_\Omega \left[v\mathbb{A}e(u)\cdot e(u) + \frac{1}{\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2 \right]\mathrm{d}x$$

where $u \in H^1(\Omega;\mathbb{R}^n)$ represents the deformation, (is not anymore BD but is instead more regular) and $v \in W^{1,\infty}(\Omega;[0,1])$ is modeling the damage of the material (*i.e.* $v=1$ at those points where no damage is present and $v=0$ in presence of fractures). As ε tends to 0 the enrgy of those points where $v \neq 1$ is blowing up at a scale ε^{-1} forcing thus the damage to concentrate on an $\mathcal{L}^n$-negligible set. Several variants of the above energy has been analyzed mostly in the scalar case (see for instance the pioneering work [AT90]). For example an approximation for a fluid driven fracture process, including a term involving an external pressure p and the divergence of the deformation u to take into account the effects of an high-pressure fluid pumped into a crack and how this contributes to the dinamic of the system, has appeared first in [XVGN17]. In collaboration with Nicolas Van Goethem (Universidade de Lisboa) we were able to validate mathematically such a model in [CVG17].

Consider now the inverse problem. Let $\mathcal{F}_\varepsilon : X \to \mathbb{R}$ be a given family of functionals (here X is any space of configurations such that $X \hookrightarrow SBD(\Omega)$) modeling a fracture problem and imagine that the sharp energy $\mathcal{F}$ (if there exists one) capturing the behavior of $\mathcal{F}_\varepsilon$ is not known. The parameter ε can represent the scale of the phenomena or just a diffusive parameter (as in homogenization processes). It is not hard to show that, under suitable assumption on $\mathcal{F}_\varepsilon$, it exists a (Γ)-limiting energy $\mathcal{F} : SBD(\Omega) \to \mathbb{R}$ but, while the energies $\mathcal{F}_\varepsilon$ comes naturally as integrals of some sort of densities over suitable domain, it is in general not clear whether the limit object still has this property or not. This is of clear importance in trying to characterize the energy $\mathcal{F}$ and in this context the issues that arise naturally are:

(I) *Can we find the energy densities of the limiting process? More precisely,*
   (Ia) *is there any integral representation for an energy $\mathcal{F} : BD(\Omega) \to \mathbb{R}$?*
   (Ib) *what are the weakest hypothesis under which question (Ia) can be answered?*

We recall that general theorems for integral representation have been developed in [DM83], [BDM85] and lately in [BFM98] where a global approach is provided. The general theory of the direct methods of Γ-convergence goes back instead to [DMM81]. In particular, the recent work [CFI17a] deals with the issue of integral representation for functional on $SBD^p(\Omega; \mathbb{R}^2)$. In addressing question (Ia) for functionals defined over $SBD^p(\Omega)$, instead of $BD(\Omega)$, we consider to keep in mind the construction developed in [CCI17]. Recent theorems on the general method on $GSBD$ have been deeply treated in the papers [Cri18] and [CC18].

An additional question, to further push our comprehension of the role of this energies in fracture mechanics is related to the differential operator $e(u)$. In many cases, as, for example, the non-interpenetrative models

$$\int_\Omega \left(e(u) - \frac{1}{n}\mathrm{div}(u) \right) \mathrm{d}x$$

the bulk part of the elastic energy is modeled by different operator rather than $e(u)$.
More in general, in the definition of the energy we can replace $e(u)$ with some differential operator $L(u)$ and consider

$$\mathcal{F}^L(u) := \int_\Omega \mathbb{A}L(u) \cdot L(u) \, \mathrm{d}x + \int_\Omega k \left(x, \frac{\mathrm{d}L^s u}{\mathrm{d}|L^s u|}(x) \right) \mathrm{d}|L^s u|(x).$$

In this case, the space of configuration should be completely rethought (we refer to [BDG17] for a detailed explanation of the space $BV^L(\Omega; \mathbb{R}^n)$ of function of bounded $L-$variation). In particular

(II) *what happen if we replace $e(u)$ with a differential operator $L(u)$? Is it physically meaningful? Can we recover all the phase-field approximation Theorems, provided the space of configuration is suitably adjusted?*

(III) *Under which assumption on L a structure theorem can be provided for the space BV^L, along the same line of the one provided for BV and BD?*

We recall the work [BDG17] where functions of bounded L-variation are treated and we stress the fact that very few things are known about this subject, making it of particular interest for further investigation. To this extent, we can find particularly useful to keep in mind the work [CFI17b], where the authors try to understand in which case an SBD function is also a BV function. Let us stress that, differently form question (I) this is a relatively new point of view in the discipline of fracture mechanics. In particular the current literature seems to present a lack on the treatment of this point and the filling of such a lack (even in part) might be a precious contribution.

1.0.2. *Work in progress - towards an answer to question I: Integral representation.* In a recent work [DPR16] due to De Philippis - Rindler the blow up of the singular part of the symmetric gradient of a BD map is characterized. Moreover, the techniques contained in the celebrated work [BFM98] seems to be the right tools to exploit such a structure theorem in order to give an answer to question I). In such a paper the authors consider an energy $\mathcal{F} : BV(\Omega) \times \mathcal{A}(\Omega) \to \mathbb{R}^+$, where $\mathcal{A}(\Omega)$ denotes the open sets of Ω. Under suitable assumption on the behavior of $\mathcal{F}$ they show that it is possible to give an integral representation of the type

$$\begin{aligned} \mathcal{F}(u, A) := &\int_A f(x, u(x), \nabla u(x)) \, \mathrm{d}x + \int_{A \cap J_u} f(x, u^+, u^-, \nu(x)) \, \mathrm{d}\mathcal{H}^{n-1}(x) \\ &+ \int_A f_\infty \left(x, u(x), \frac{\mathrm{d}C(u)}{\mathrm{d}|C(u)|}(x) \right) \mathrm{d}|C(u)| \end{aligned} \tag{1.2}$$

where $C(u)$ denotes the cantor part of Du, f_∞ is the recession function of f and f, g are functions defined throughout a blow-up procedure. The work heavily rely, for what concerns the cantor

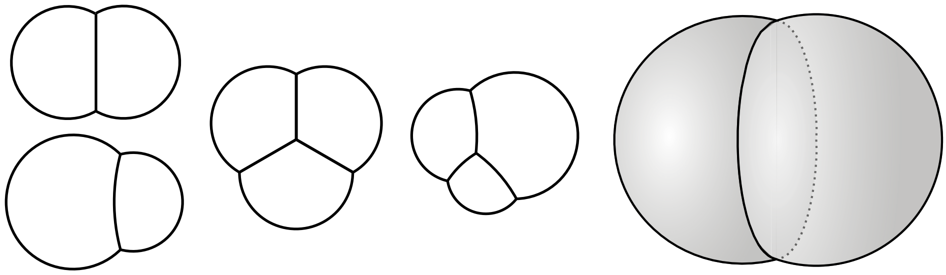

FIGURE 2.1. Some examples of perimeter minimizers N-clusters for $N = 2, 3$ in dimension $n = 2$ and $n = 3$. The 2-clusters on the left are, respectively, the minimizer for the problem (2.1) with equal-volume (equal-area) chambers $m_1 = m_2$ and when different volumes $m_1 \neq m_2$ have been assigned. The same situation is the central one, for $N = 3$, while the right-hand picture is the perimeter minimizer 2-cluster for equal-volume chambers in $\mathbb{R}^3$.

part $C(u)$, on the rank-one property of BV function proven by Alberti in [Alb93]. The 2016 breakthrough in [DPR16] yields as a consequence, among a series of interesting facts, that

$$\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x) = a(x) \odot b(x)$$

where $E^c u$ denotes the Cantor part of the distributional symmetric gradient. This fact opens the door to an extension of the integral representation Theorem for energy on $BD(\Omega) \times \mathcal{A}(\Omega)$ as (1.2). It is worth noticing that such a Theorem has been obtained for energies defined on SBD in [ET01], by exploiting the technique introduced in [BFM98]. The result contained in [ET01] is not taking into account the Cantor part since there were no characterization available for the singular measure at the time: gap that can now be filled due to the recent development in this field.

2. Planar minimal N-Clusters: grain boundaries as isoperimetric objects

Given a natural number N and $m_1, \ldots, m_N$ real positive number we look for a family of N essentially disjoint (disjoint up to a set of measure zero) set $\mathcal{E} = \{\mathcal{E}(1), \ldots, \mathcal{E}(N)\}$ having volume $|\mathcal{E}(i)| = m_i$ and minimizing the total perimeter. Any family $\mathcal{F} = \{\mathcal{F}(1), \ldots, \mathcal{F}(N)\}$ of N essential disjoint sets is also called an N-cluster and we sometimes refer to the set $\mathcal{F}(i)$ as the chambers of the cluster. In order not to trivialize the problem, it is usually adopted the convention that any common interface between two sets of the family must be counted once. In particular, the perimeter of a N-cluster $\mathcal{E}$ is defined as

$$P(\mathcal{E}) = \sum_{i=0}^{N} \frac{P(\mathcal{E}(i))}{2}$$

where we are defining the *external chamber* $\mathcal{E}(0)$ to be

$$\mathcal{E}(0) = \left(\bigcup_{i=1}^{N} \mathcal{E}(i) \right)^c .$$

With this notation, the multi-chamber isoperimetric problem (a natural generalization of the classical isoperimetric problem) consists in finding the minimizers (whenever they exists) for the infimum problem:

$$\gamma_N(m_1, \ldots, m_N) := \inf \{P(\mathcal{E}) \mid \quad |\mathcal{E}(i)| = m_i, \mathcal{E} \text{ is an } N\text{-cluster}\} . \tag{2.1}$$

The N-clusters $\mathcal{E}$ attaining a minimum in (2.1) for some values $m_1, \ldots, m_N$ are called isoperimetric N-clusters or perimeter minimizer N-clusters.

2.0.1. *State of art and main questions.* The existence and regularity of isoperimetric N-clusters for any given volumes and in any ambient space dimension has been proven by Almgren in [Alm76] (see also [Mag12]). Since the study of these objects for any fixed N turns out to be a very hard task (so far the solution is only known for $N = 2$ in any dimension [FAB+93], [Rei08], for $N = 3$ in dimension 2 [Wic02], and some recent progress has been made in dimension 2 for $N = 4$ [PT16], see figure 2.0.1 for some example) in my Ph.D Thesis [Car16] together my advisors Giovanni Alberti and Francesco Maggi we tried to develop an asymptotic analysis in N of the problem (2.1) in various frameworks. From the pioneering work of Fejes Toth [Tót64] and the more recent and general one by Hales [Hal01] it is a well known fact that the hexagonal tiling provide the best way to enclose and separate unit-area regions with the least amount of perimeter. As a consequence of the results achieved in [CM16], in collaboration with Francesco Maggi, and in [Car16], together with Giovanni Alberti as a part of my Ph.D thesis, it is readable that the interior chambers of a perimeter minimizing planar N-clusters will converge (in average) in L^1 to regular hexagons. In the interests of a complete description of the behavior of planar minimal cluster $\mathcal{E}_N$ it is useful to focus our attention on the carriage of the external chamber $\mathcal{E}_N(0)$ (or $\mathcal{E}_N(0)^c$ which makes no difference). Concerning this question we have the following conjecture by Heppes and Morgan appearing in [HM05].

Conjecture 2.1. *Let $\{\mathcal{E}_N(0)\}_{N\in\mathbb{N}}$ be a sequence of external chambers of perimeter minimizing N-clusters $\mathcal{E}_N$ having unit-area chambers. Then $\frac{\mathcal{E}_N(0)^c}{\sqrt{N}} \to B$ in L^1 where B is the unit-area ball.*

An interesting way to approach this problem might be the study of the functional,

$$G_\delta(E) := \inf\left\{P(\mathcal{E}) \;\middle|\; \mathcal{E} \text{ N-cluster with } \mathcal{E}(0)^c \supset E,\ |\mathcal{E}(i)| = \delta\right\}.$$

We realize immediately that Hales' Theorem and a comparison argument say that

$$\frac{P(H)}{2}|E| < G_\delta(E) \leq \frac{P(H)}{2}|E| + c\frac{\sqrt{|E|}}{\sqrt{\delta}}, \tag{2.2}$$

so if we were able to compute

$$F := \Gamma - \lim_{\delta\to 0^+} \sqrt{\delta}G_\delta,$$

F can provide the way to solve the conjecture. If F turns out to be a multiple of the perimeter functional then the Heppes-Morgan conjecture can be proved by just observing that, whenever $\mathcal{E}_N$ is a unit-area perimeter-minimizing cluster, we immediately have that $\frac{\mathcal{E}_N(0)^c}{\sqrt{N}}$ is a minimum for $G_{\frac{1}{N}}$. Thus this fact will imply that any accumulation point E of $\left\{\frac{\mathcal{E}_N(0)^c}{\sqrt{N}}\right\}$ is a minimizer for the perimeter, which means: E is a ball.

Unfortunately the computation of F is really an hard question. Moreover there is a good chance that in this framework we are not taking into account the important fact that the Γ-limit could be strictly related to the behavior of the interior chambers. Even if we assume that in some part

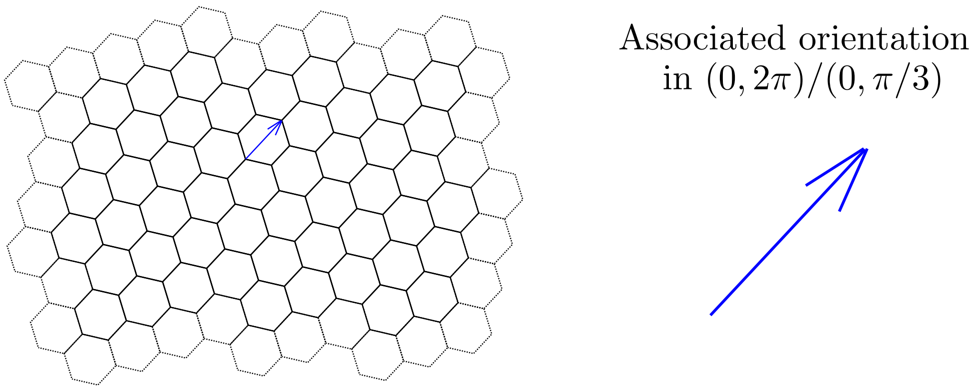

FIGURE 2.2. How to define, in a very natural way, an orientation for an hexagonal tiling.

of $\mathcal{E}_N(0)^c$ the interior chambers are converging to some hexagonal tilings there is no reasons such tilings should have the same, orientation in the sense of figure 2.2. It could happen that in a certain cube $Q_1 \subset \mathcal{E}_N(0)^c$ there is a tiling with an orientation v and far away in some other cube $Q_2 \subset \mathcal{E}_N(0)^c$ we see another hexagonal tiling with a different orientation $w \neq v$. If this is the case, somewhere in the cluster the following problem is solved.

I) *Given two grid $\mathcal{H}_1, \mathcal{H}_2$ with different orientation, it is possible to find a unit-area cluster $\mathcal{F}$ having the following property*

$$I.1) \qquad G = \mathcal{H}_1(0)^c \cup \mathcal{F}(0)^c \cup \mathcal{H}_2(0)^c \text{ is connected (indecomposable)}$$

$$I.2) \qquad \text{defined } \mathcal{G} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{F}\} \text{ then } P(\mathcal{G}; Q) \approx \frac{P(H)}{2}|Q| + c\sqrt{N}.$$

In other word given two grid having different orientation, can we find a *binding* $\mathcal{F}$ connecting them and with local perimeter close to the ground state, up to a lower order term (see figure 2.3)? With a positive answer to this question in hands, we would be able to prove the conjecture by showing that the gamma limit is actually a multiple of the perimeter with a constructive proof.
An additional step forward towards the comprension would be to link such orientations (that might also not be clearly defined for general N-cluster) to the perimeter length. In particular

II) *Is there a way to assign an orientation to almost-minimizing N-cluster that is linked to the perimeter in the following sense: given $u_N : \mathbb{R}^2 \to \mathbb{R}^2$ the map catching the orientation of $\mathcal{E}_N$ and given that $\sqrt{P(\mathcal{E}_N)}N \to P(H)/2$ then $u_N \to u_{\mathcal{H}}$ where $u_{\mathcal{H}} : \mathbb{R}^2 \to \mathbb{R}^2$ is a map catching the orientation of an hexagonal tiling?*

Let me point out that, a priori, the sequence $\left\{\frac{\mathcal{E}_N(0)^c}{\sqrt{N}}\right\}$ could have more than one limit point and that they can be different depending on the subsequence we are looking at. For example if N_k is the sequence of the hexagonal numbers (the numbers for which a big hexagon of hexagons can be built) it seems reasonable to presume that the eventual limit in this case is H:

$$\frac{\mathcal{E}_{N_k}(0)^c}{\sqrt{N_k}} \to H.$$

Observe that for $\mathcal{E}_{N_k}$ the rigidity of the interior chambers is in force and only one orientation is present in $\mathcal{E}_{N_k}(0)^c$.

A recent work along the same line depicted here has been recently releases [DLNP18], even if in a slightly different context.

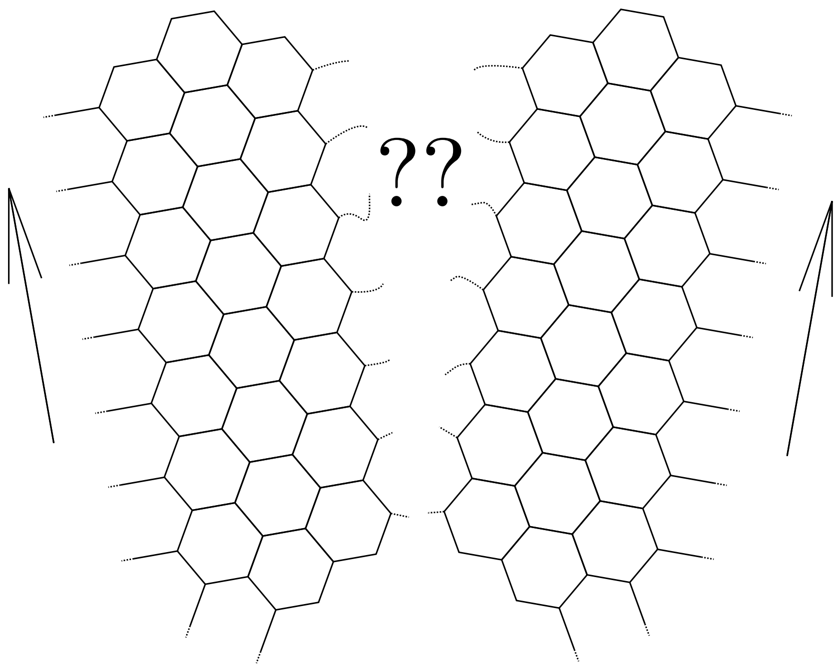

FIGURE 2.3. Is there a way to connect with a cluster having unit-area chambers two hexagonal grid with different orientation without increase the local perimeter too much??

2.0.2. *Work in progress - Compactness of orientation from perimeter bounds (with Giovanni Alberti and Giacomo Del Nin).* With Giovanni Alberti and Giacomo del Nin we started to discuss several ideas that might provide an answer to question II. By starting from a tiling of the Torus with the same topology of the hexagonal one (chambers of six edges and singularities made by three edges joining together into a point) we can define a map in the following way. Given a chamber $\mathcal{E}_N(i)$ we first define $u_i : \partial\mathcal{E}_N(i) \to \partial H$ that maps the boundary on the boundary (here H is a reference hexagon) and then we consider its harmonic extension defined on the whole chamber. We then glue together all the maps. The interesting fact is that the following estimate on the map u can be provided

$$\int_{\mathcal{T}} d(\nabla u(x), SO(2))^2 \,\mathrm{d}x \leq C[P(\mathcal{E}_N) - P(\mathcal{H})]$$

for a universal constant C that does not depend on N. The rigidity Theorem by Friesecke, James and Müller [FJM02] ensures then that, for some rotation $R \in SO(2)$,

$$\|\nabla u - R\|_{L^2} \leq C[P(\mathcal{E}_N) - P(\mathcal{H})].$$

This means that we can give an estimate on the gradient of the map in term of the perimeter deficit. This is just the beginning of our work and the real challenge would be now to develop such construction also on tilings with other and more complicated topology. An interesting connection with dislocation theory (and also linear elasticity) seems to arise from this approach when we treat topological defects (pentagon, octagon etc.) of tilings. Let us recall that a similar approach has been succesfully adopted in [The06] to treat a crystallization model.

3. Additional ongoing projects

3.0.1. *Relaxation of energies with densities (with Riccardo Cristoferi).* In the work [CCD18] the energy

$$E(\Omega, u) := \int_{\partial\Omega} \psi(u) \,\mathrm{d}\mathcal{H}^{n-1}(x) \tag{3.1}$$

is studied. As a consequence of such work we achieve the Γ-convergence of the energies

$$E_\varepsilon(\phi, u) := \int_{\mathbb{R}^n} \psi(u) \left(\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}W(\phi)\right) \mathrm{d}x$$

(under the topology considered on the space of configuration) to the relaxation of E. A general question that arises under the light of this recent approach proposed in [CCD18] is the following. Let $\mathcal{F} : BV(\Omega) \times \mathcal{A}(\Omega) \to \mathbb{R}^+$ be an energy such that $\mathcal{F}(\phi; \cdot) := \mu_\phi(\cdot)$ is a Radon measure for every $\phi \in BV(\Omega)$ and such that $\mathcal{F}(\cdot; A)$ is lower semi-continuous with respect to the L^1 convergence. Consider then $\mathcal{G}^{\mathcal{F}} : BV(\Omega) \times \mathcal{M}_b(\Omega) \to \mathbb{R}$, where $\mathcal{M}_b(\Omega)$ is the family of all the Radon measures on Ω, be defined as

$$\mathcal{G}^{\mathcal{F}}(\phi, \mu) \begin{cases} \displaystyle\int_\Omega \psi(u(x)) \,\mathrm{d}\mu_\phi(x) & \text{if } \mu = u\mu_\phi,\ u \in L^1(\Omega, \mu_\phi) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2}$$

for a function ψ. Notice that this process allows us to consider a density u weighted on those part of Ω where ϕ puts non trivial energy. It represents a generalization of the case $\mathcal{F}(\mathbb{1}_E, A) := P(E; A)$ yielding (3.1). Along the line of the work [CCD18] we can ask what is the relaxation of $\mathcal{G}^{\mathcal{F}}$ under the $L^1 \times w^*$ topology. How the association $\mathcal{F} \mapsto \mathcal{G}^{\mathcal{F}}$ behaves under Γ-convergence? Namely, if $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}$ then, can we deduce that $\mathcal{G}^{\mathcal{F}_\varepsilon}$ Γ-converges to $\mathcal{G}$?

Together with Riccardo Cristoferi we are trying to address such question starting from the general approach suggested by the *general method for relaxation* contained in [BFM98]. In particular we are studying the cell problem

$$m((\phi, \nu); Q) := \inf\left\{ \int_Q \psi(u) \,\mathrm{d}\mu_v(x) \;\middle|\; \begin{array}{c} v \in BV(Q),\ v = \phi \text{ on } \partial Q \\ u \in L^1(Q, \mu_v),\ u\mu_v(Q) = \nu(Q). \end{array} \right\}.$$

as a starting point in order to relax (3.2), and in particular the comprhension of the quantity

$$\lim_{\varepsilon\to 0}\frac{m((\phi,\nu);Q(x_0,\varepsilon)}{\mu_\phi(Q(x_0,\varepsilon))}$$

is a crucial step in running the technique introduced in [BFM98]. Let us underline that such a general approach seems to be a novelty for what concerns energy with two variable of the above shapes. Several energies, beyond the ones treated in [RV06], can be defined starting from (3.2).

3.0.2. *Regularity for the fractional isoperimetric function (with Matteo Rinaldi and Riccardo Scala).* In collaboration with Matteo Rinaldi, a former CMU Ph.D. student, and Riccardo Scala a post-doc at the University of Lisbon, we are studying the regularity of the fractional isoperimetric function. The fractional perimeter has received increased attention from the mathematical community since it has been introduce by Caffarelli, Roquejoffre and Savin in [CRS10] in 2010. Given a generic set E and $s\in(0,1)$, the s-fractional perimeter of E can be defined as

$$P_s(E):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\mathbb{1}_E(x)-\mathbb{1}_E(y)|}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y=\int_E\int_{E^c}\frac{1}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.3}$$

namely the fractional s-Sobolev semi-norm of the characteristic function $\mathbb{1}_E$ of E. In particular, given an open smooth set Ω and $m\in(0,|\Omega|)$, it is possible to consider the volume constrained fractional isoperimetric function:

$$I_\Omega(m;s):=\inf\left\{P_s(E)\mid |E|=m,\quad E\subset\subset\Omega\right\}. \tag{3.4}$$

Our principal aim is to study the regularity of the function $I_\Omega(m;s)$ for a fixed $s\in(0,1)$ with respect to the variable m. In the classical case it has been shown that if Ω is $C^{2,\sigma}$ then the isoperimetric function is C^1 almost everywhere on $(0,|\Omega|)$, [MR15, Remark 4.4]. The authors used this result to estimate the rate of convergence of a phase field approximation of the non local Allen-Cahn equation. Regularity of the classical isoperimetric function has been used also in [LM16]. The recent tools delevoped in [CG10] provide a way to deduce the $C^{1,\gamma}$ regularity for the minimizers of problem 3.4 which is a first step toward proving the regularity of the function $I_\Omega(m;s)$. Let me underline that a useful set of tools to approach this problem is contained in the discussion of the non-local capillarity problem treated in [MV17].

3.0.3. *Average distance problem (with Xin Yang Lu and Ihsan Topaloglu).* Together with doctor Lu and professor Topaloglu we are trying to understand the minimizers of an attractive-repulsive energy of the type

$$AD(E):=\int_{E\times E}\left[\frac{1}{p}|x-y|^p-\frac{1}{q}|x-y|^{-q}\right]\,\mathrm{d}x\,\mathrm{d}y.$$

The function $u(r):=\frac{1}{p}r^p-\frac{1}{q}r^{-q}$ is strictly convex in r and this is enough (even if is not straight-forward) to obtain that by performing a Steiner symmetrization the energy decreases, argument that applies to any energy whose integrand is the composition of a convex function with $|x-y|$. We are currently trying to show that the balls are the only minimizers among all the Borel sets with fixed volume. This problem seems to be well studied in dimension $n=2$ for $u(r)=|r|$ see [Dun97] where also the biological motivation is discussed) and seems that no results are available in bigger dimensions. Let us also recall that the similar energy for Radon measure

$$E(\mu):=\int_{\mathbb{R}^n\times\mathbb{R}^n}W(x-y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

for particular potential W it has been widely studied, see for example [BCLR13] and [CFP17].

3.0.4. *On the contact surface of Cheeger sets (with Gian Paolo Leonardi - Università degli studi di Modena).* Given a smooth open set Ω the Cheeger constant of Ω (introduced first in [Che70]) is defined to be

$$h(\Omega)=\inf\left\{\frac{P(F)}{|F|}\;\middle|\;F\subset\subset\Omega\right\},$$

where $P(F)$ denotes the distributional perimeter of the set (see [Mag12]) and $|F|$ is the Lebesgue measure of F. Any set $F\subseteq\Omega$ such that $\frac{P(F)}{|F|}=h(\Omega)$ is called a Cheeger set of Ω. It is well-known that for any Cheeger set F of Ω the interior boundary $\partial F\cap A$ is a $C^{1,\alpha}$-hypersurface with constant mean curvature $H=\frac{h(E)}{n-1}$. We refer to [Leo15], [Par11], for two exhaustive surveys on the Cheeger problem. In [LP14] the authors show that F meets the boundary of Ω tangentially and $\nu_E(x) = \nu_\Omega(x)$ for any $x \in \partial F \cap \partial\Omega$, but no results are known about the size of the contact set $\partial F \cap \partial\Omega$. The question we are currently addressing is under which hypothesis is it possible to infer that $\mathcal{H}^{n-1}(\partial F \cap \partial\Omega) > 0$ for a given Cheeger set F of Ω. The strategy is, under some suitable assumption, to try to derive a contradiction on the shape of an eventual Cheeger set such that $\mathcal{H}^{n-1}(\partial F \cap \partial\Omega) = 0$ by exploiting Alexandrov's Theorem [Ale62]. This question is linked to the results describing the structure of Cheeger N-Clusters [Car17] that I have treated in my Ph.D. thesis.

4. Objectives

One of the biggest challenge I would like to face throughout this project is to get closer and build a strong link with the community of applied mathematician and engineers. I think that, in order to push forward my comprehension on these topics and with the aim of further extending the results here conjectured it is of crucial importance, nonetheless incredibly stimulating for me, to talk with people from different fields. My actual ongoing projects shows that I am capable of working with several co-authors in a constructive and efficient way.

I would really like to start a dialogue with numerical analyst in order to explore from a computational point of view several ideas and conjectured that arise naturally when looking at the problem exposed in Sections 1 and 2.

For what concern results about the first two section, instead, in the next two year I expect the following.

a) Question (I) in Subsection 1.0.1 is already on a good road with my co-authors (see 1.0.2) and we expect to find sharp (and physically relevant) hypothesis that can guarantee an integral representation for functional $\mathcal{F} : BD(\Omega;\mathbb{R}^n) \to \mathbb{R}$;
b) Concerning question (II) the aim is to extend the phase-field approximation to possibly more general differential operator, but is not clear what hypothesis might be needed on L in order to import the existing tools. We do think that some interesting cases (as the deviatoric operator $e(\cdot) - 1/n\mathrm{div}(\cdot)$) could be treated;
c) I expect to understand in deep what are the conditions under which question (III) can be treated. This might represent the hardest question and in fact, we do not really know how delicate it is. As pointed out in [BDG17], even classical instrument as the existence of L^1 traces, might fail already in dimension $n = 2$;
d) I do not think that a binding of the type requested in question I) in 2.0.1 does exist and I expect to work in this direction in order to prove some sort of rigidity Theorem for almost minimal tilings;
e) We expect to give a positive answer to question II) in 2.0.1 and to define a candidate Γ-limit on the space of configuration;

The Section 3 instead does not contains problem that can be inserted in a global picture yet and we would like to build around them a more solid framework in the next two years. Such problems are all in progress and we expect to close them in short time.

5. Methodology

My previous experience at CMU, collaborating with Riccardo Cristoferi and Laurent Dietrich on a problem from materials science gave me all the tools I need to treat the Γ-convergence and approximation questions stated here. In particular in [CCD18], we dealt with a functional defined over (we refer also to the attachment for additional details) we dealt with the relaxation of a functional defined over $BV(\Omega;\{0,1\}) \times \mathcal{M}$ where $\mathcal{M}$ is the space of positive Radon measure. The techniques we studied and the background we developed, since relaxation and integral representation are interconnected issues, gave me the required skills to consider more general problems as the one stated in question (I), (II) and (III) of Section 1.0.1.

Additional methods in my backrgound come from the work I did with Dejan Slêpcev and Antonin Chambolle on a topic involving a discrete version of the Mumford-Shah functional which has the same structure of the fracture models proposed here, but in the scalar case (the work is still in preparation). The techniques developed involves, as shown in [GTS16], an interesting application of optimal transportation theory that seems to be particularly appropriate for numerical approximation and analysis of energies defined on cloud points.

The work I am doing here at the Universidade de Ciências on the model proposed in [XVGN17], together with Nicolas Van Goethem, put me close to all the question concerning BD and SBD functions giving me a strong picture of all the main techniques currently available, combined with a solid background in geometric measure theory and phase-field approximation.

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