Curriculum vitæ,
Research and teaching statement

July 19, 2024

Théophile Dolmaire

[redacted]

After a short curriculum vitæ, presented from page 2 to 8, theme by theme (list of qualifications, list of publications and pre-prints, list of talks, teaching and advising experience), a detailed research statement is presented, starting page 9. The three first sections are devoted to recall the classical results concerning the Boltzmann equation and its derivation, in order to present the background and the context that motivated my researches.
The fourth section contains a report on my research concerning the derivation of the Boltzmann equation, whereas in the fifth section my works concerning the systems of inelastic particles are presented. The sixth section deals with my ongoing projects.
Finally, the seventh section is a short comment about my different teaching experiences, together with some perspectives and projects.

Table of content

1 Introduction to the derivation of the Boltzmann equation **9**
- 1.1 From the beginning of kinetic theory to the formal derivation of the Boltzmann equation 9
- 1.2 Properties of the Boltzmann equation . . . 11
- 1.3 Loschmidt's and Zermelo's paradoxes, discussion of the model . . . 11
- 1.4 Granular gases : kinetic theory for dissipative particle systems . . . 12

2 Rigorous derivation, Boltzmann-Grad limit and Lanford's theorem **13**
- 2.1 BBGKY hierarchy and Boltzmann-Grad limit . . . 13
- 2.2 Rigorous derivation I: solving the hierarchies. . . . 15
- 2.3 Rigorous derivation II: convergence of the BBGKY hierarchy . . . 17

3 Contributions to the problem of the derivation of the Boltzmann equation **21**
- 3.1 Question of the appropriate functional spaces to solve the hierarchies . . . 21
- 3.2 Rigorous derivation of the Boltzmann equation in a domain with boundary: the case of the half-space . . . 23
- 3.3 Towards domains with a more general geometry: partial results in the case of the disk 25

4 Contributions to the theory of inelastic particles **27**
- 4.1 Study of the inelastic collapse of 3 particles in dimension $d \geq 2$: classification of singularities and explicit construction of collapses . . . 27
- 4.2 An example of an inelastic particle system whose flow preserves the measure in the phase space, but whose kinetic energy is not conserved . . . 30

5 Ongoing projects, and perspectives **31**
- 5.1 Ongoing projects and collaborations . . . 31
- 5.2 Insertion of my research themes into the University . . . 31

6 Comments on past teaching experiences and perspectives **32**
- 6.1 Past teaching experiences . . . 32
- 6.2 Perspectives about teaching . . . 33

List of qualifications

Théophile Dolmaire

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French, born the 27/09/1993.

I dispose of all of my civil rights as a French citizen, and I have an adequate knowledge of the Italian language.

The following persons can support and recommend my application.

- Laurent Desvillettes (Université Paris Cité, Ph.D advisor), desvillettes@imj-prg.fr
- Isabelle Gallagher (ENS Paris, Ph.D advisor), gallagher@imj-prg.fr
- Chiara Saffirio (Universität Basel), chiara.saffirio@unibas.ch
- Juan Velázquez (Universität Bonn), velazquez@iam.uni-bonn.de

Current and previous positions

2022– **Post-doctoral fellow at the University of Bonn, Germany**
Study of the systems of inelastic particles, under the supervision of Juan Velázquez.

2019–2022 **Post-doctoral fellow at the University of Basel, Switzerland**
Study of the derivation of the Boltzmann equation in domains with complicated geometry, under the supervision of Chiara Saffirio.

2016–2019 **Post-doctoral fellow at the University Paris Diderot**
Study of the Lanford theorem concerning the derivation of the Boltzmann equation, and adaptation to domains with boundary.

Studies

2016–2019 **Université Paris Diderot (Paris VII)**, Paris, France,
Ph.D in Mathematics. Titel : *Mathematical derivation of the Boltzmann equation with boundary condition*, under the supervision of Laurent Desvillettes and Isabelle Gallagher.

2015–2016 **Université Paris Diderot (Paris VII)**, Paris, France,
Master 2 of Mathematics, specialization "Fundamental Mathematics". Mention "Très bien" ("Very Good", above 16/20).

2015 **Concours de l'Agrégation externe de Mathématiques**,
(French competitive examination, providing the degree to teach in high schools and Universities), prepared at the ENS Cachan.

2012–2013 **Université Paris Diderot (Paris VII)**, Paris, France,
Bachelor of Mathematics, specialization "Fundamental Mathematics". Mention "Très bien" ("Very Good", above 16/20).

2012–2016 **École Normale Supérieure de Cachan**, Cachan, France,
Élève normalien, Mathematics department.

Other scientific activities

2020 **Reporter of the workshop** "Classical and Quantum Mechanical Models of Many-Particle Systems" (Oberwolfach, december 2020).

2019– **Referee** for the journals *Archive for Rational Mechanics and Analysis*, and *Communications in Mathematical Physics*.

Languages

French native speaker.

English fluent.

German intermediate (A2.2).

Italian basic notions.

Programming

MATLAB experimented.

List of publications and pre-prints

Théophile Dolmaire

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French, born the 27/09/1993.

Themes of research

My themes of research belong to the field of kinetic equations. I worked on the rigorous derivation of the Boltzmann equation, and I studied in particular the derivation of the classical equation (the fully non-linear Boltzmann equation, in the non-relativistic case, for the hard sphere collision kernel) in the case of domains with boundaries.
Besides, I worked also on interacting particles systems. I studied especially the dynamics of the systems of inelastic hard spheres, in order to extend the results of rigorous derivation of Lanford to granular gases.

Publications and pre-prints

2024 **1.** (with Juan Velázquez), **Properties of some dynamical systems for three collapsing inelastic particles**, arXiv:2403.16905.
We continue the study of the systems of three inelastic particles started in the article "Collapse of inelastic hard spheres [...]". In particular, we show that the study can be formally reduced to a two-dimensional dynamical system, and we study rigorously and numerically such a system.

2024 **2.** (with Juan Velázquez), **A particle model that conserves the measure in the phase space, but does not conserve the kinetic energy**, arXiv:2403.02162.
We study a system of inelastic particles in dimension 2 (where a fixed quantity of kinetic energy is lost during each collision). We show in particular that the flow of the dynamics of this system conserves the measure in the phase space, although the kinetic energy is not conserved. We deduce that the dynamics of such a system is globally well-posed for almost every initial datum (Alexander's theorem). To the best of our knowledge, this is the first version of an Alexander's theorem for inelastic particles.

2024 **3.** (with Juan Velázquez), **Collapse of inelastic hard spheres in dimension $d \geq 2$**, arXiv:2402.13803v2.
We study a system of three inelastic particles, with fixed restitution coefficient, in dimension $d \geq 2$. We show in particular that when the inelastic collapse takes place, the particles can collide according to only two possible orders of collisions, that are periodic. We deduce important information on the system at the time of collapse. We build explicit initial data that lead to the inelastic collapse, in a stable way, and such that the geometry of the system at the time of collapse can be chosen a priori, with an arbitrary precision. To the best of our knowledge, this is the first construction of a stable inelastic collapse in dimension $d \geq 2$.

2023 **4. About Lanford's theorem in the half-space with specular reflection**, in *Kinetic and Related Models*, **16**:2 (04/2023).
We perform the rigorous derivation of the Boltzmann equation starting from a system of hard spheres, in the case of the half-space, where the boundary condition is the specular reflection. This is the first rigorous derivation of the Boltzmann equation in a domain with boundary, with a quantified rate of convergence.

Invitations

November 2023 — **LJLL, Sorbonne Université**, in order to collaborate with Nathalie Ayi.

Talks in conferences or seminars

September 2023 — **MFO, Oberwolfach, Germany**. Workshop "Classical and Quantum Mechanical Models of Many-Particle Systems".

July 2022 — **Villa Maria Seminar, Universität Bonn, Germany**. Group seminar of the Ph.D students.

May, October 2022 — **IAM, Universität Bonn, Germany**. Presentation in three parts, in the group seminar of the Functional Analysis team.

November 2021 — **Universität Basel, Switzerland**. Inaugural presentation of the Kinetic Theory Seminar, co-organized by the Universität Basel and the ETH Zürich.

September 2021 — **Ravello, Italy**. XLVI Summer School on Mathematical Physics.

June 2021 — **IAM, Universität Bonn, Germany**. Oberseminar Analysis.

January 2021 — **POSTECH, Pohang, South Korea**. Analysis seminar (online, mini-lecture) of the PDE group, POSTECH Mathematics Institute.

December 2019 — **Centre for Mathematical Sciences, University of Cambridge, England**. MAFRAN days.

November 2019 — **Institut de Mathématiques de Bordeaux, Université de Bordeaux Talence, France**. "Inaugural France-Korea Conference on Algebraic Geometry, Number Theory, and Partial Differential Equations".

April 2018 — **CEREMADE, Université Paris Dauphine, France**. Group seminar of the Ph.D students.

February 2018 — **Université Pierre et Marie Curie, Paris, France**. Group seminar of the Ph.D students of the lab. IMJ-PRG.

December 2017 — **Université Pierre et Marie Curie, Paris, France**. Group seminar of the Ph.D students in PDE.

May 2017 — **ENS Ulm, Paris, France**. Group seminar of the Ph.D students in PDE.

Teaching and advising experience

Théophile Dolmaire

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French, born the 27/09/1993.

I had teaching experiences, as a teaching assistant as well as a lecturer, in France, Switzerland and Germany, in French and English, at the undergraduate and graduate level. The following persons can evaluate my implication and my methods regarding teaching:

- Davide Barilari (Università di Padova) for the period 2016–2019, davide.barilari@unipd.it,
- Chiara Saffirio and Gianluca Crippa (Universität Basel) for the period 2019–2022, chiara.saffirio@unibas.ch and gianluca.crippa@unibas.ch,
- Juan Velázquez (Universität Bonn) for the teaching and advising experience, for the period 2022 à 2024, velazquez@iam.uni-bonn.de.

Teaching at the undergraduate level

2019–2022 **Teaching assistant at the University of Basel**

- **Iterative Verfahren der Numerik**. 2019-2020 (Semester 1). Introduction to numerical methods applied to optimization. Lecture: Marcus Grote.
- **Reelle Analysis**. 2021-2022 (Semester 1). Measure theory and integration. Lecturer: Chiara Saffirio.

2016–2019 **Teaching assistant at the University Paris Diderot**

- **Exercise classes for biologists**. 2016–2017 (Semester 1). Lecturer: Tamara Servi. Introduction to analysis.
- **Exercise classes of elementary analysis and algebra**. 2016–2017 (Semester 2). Lecturer: Fabrice Vandebrouck.
- **Advanced exercises**. 2017–2018 (Semester 2), and 2018–2019 (Semester 2). Exercise classes for motivated students, with the aim of deepening the concepts.
- **Exercises classes on ODE**. 2017–2018 (Semester 2), and 2018–2019 (Semester 2). Lecturer: Davide Barilari.

Teaching at the graduate level

2023–2024 **Lecturer at the University of Bonn**
Introduction to the Boltzmann equation. Semester 1.

2022–2023 **Organizer of a "graduate seminar" at the University of Bonn**. Semester 1. Study and presentation of recent articles around the topic "The Boltzmann equation and its hydrodynamic limits".

2020–2021 **Lecturer at the University of Basel**
Kinetic Equations 2020-2021 (Semester 2). Lecturer of the second half of the lecture, the first being taught by Chiara Saffirio. Introduction to the Vlasov and Boltzmann equations.

2019 – 2021 **Teaching assistant at the University of Basel**

- **Analysis II**. 2019-2020 (Semester 2). Lecturer: Enno Lenzmann.
- **Differential equations and Sobolev spaces**. 2020-2021 (Semester 1). Lecturer: Gianluca Crippa.

Advising experience

2022–2024 **Advisor of master theses at the University of Bonn**.

- **Master thesis of Eleni Hübner-Rosenau**, 2022–2023, "Some Problems in Particle Systems: Inelastic Hard Spheres". Eleni Hübner-Rosenau defended her master thesis in november 2023, and she started a Ph.D. at the University of Regensburg.
- **Master thesis of Daniel Happ**, 2023–, co-advised with Eugenia Franco, "Eigenelements of a general aggregation-fragmentation model".

Research and teaching statement

1 Introduction to the derivation of the Boltzmann equation

1.1 From the beginning of kinetic theory to the formal derivation of the Boltzmann equation

Kinetic theory finds its origins in the works of Bernoulli [4], who sought from 1738 to describe gases by studying the movements of their elementary components. An important step is crossed by Maxwell ([36], [37]) when he obtained in 1867 the expression of the velocity distribution of a gas at thermal equilibrium: the celebrated Maxwellian function. Inspired by this work, Boltzmann investigated the general case, and obtained the equation for the evolution of the particle distribution of a gas, out of thermodynamic equilibrium [7]. Boltzmann studied the evolution of the following quantity:

$$f = f(t, x, v), \tag{1}$$

which represents the number of particles of the fluid that is considered (or its density, up to renormalize), lying at time t, at a position that belongs to $x + dx$, and moving with a velocity that belongs to $v + dv$, where dx and dv are two elementary volumes. Such an equation, depending on the variables t, x and v, is called a *kinetic equation*.

Let us describe with more details the approach that allowed Boltzmann to obtain his equations (we refer to [53] for more details), which is instructive and exhibit the main difficulties that one encounters when one study the Boltzmann equation.
Firstly, if we consider a very large number of particles, which evolve according to Newton's laws in Euclidean space $\mathbb{R}^d$ ($d \geq 2$) and which are not subject to any force (inertial movement), then it is clear that the particle density satisfies the *free transport equation*:

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = 0, \tag{2}$$

Of course, this naive model oversimplifies the dynamics of the particles. If we now want to take particle interactions into account, we can describe the system using the *hard sphere model*: let us assume that the particles are spheres of the same radius and mass, moving at constant speed when they are not in contact, and collide elastically when they collide. More precisely, we will assume here that momentum and kinetic energy are conserved during collisions. These conservation laws mean that there exists $\omega \in \mathbb{S}^{d-1}$ such that:

$$\left\{ \begin{array}{rcl} v' & = & v - \left((v - v_*) \cdot \omega \right) \omega, \\ v'_* & = & v_* + \left((v - v_*) \cdot \omega \right) \omega, \end{array} \right. \tag{3}$$

and in the case of the specular reflection (that is, in the case which corresponds to the physical collision between two billiard balls), the physical choice consists in taking:

$$\omega = \frac{(x_2 - x_1)}{|x_2 - x_1|}, \tag{4}$$

where x_1 and x_2 are the respective positions, at the time of collision, of the two spheres involves in such a collision. The unitary vector ω is called the *impact parameter*, which is, in the case of the hard spheres, orientated along the line joining the centers of the two colliding particles.
In order to take into account the collisions, that are also able to modify the quantity $f(t, x, v)$, the Boltzmann equation has then to be rewritten as follows:

$$\partial_t f + v \cdot \nabla_x f = Q, \tag{5}$$

where Q is the *collision term*. We will assume that the effects of collisions are localized in time and space, that is to say, only particles close to the position x at time t are likely to modify the velocities of the particles which are found at time t in the volume $x + dx$ (this is a fundamental difference from the models behind other kinetic equations, such as the Vlasov equation, for which particles interact via gravitational or electrostatic, potentials, over long distances). Therefore, we obtain:

$$Q(t,x,v) = Q\left(S(t,x)\right)(t,x,v), \tag{6}$$

where S described the system of particles, locally at time t, in the neighbourhood of x. Let us assume now that the gas is *dilute*, that is, with a very low density. In this case, binary collisions are much more likely than any other type of collisions involving more than three particles. As a result, Q will depend only on the density $f^{(2)}$ of the pairs of particles of the system:

$$f^{(2)}(t,x_1,v_1,x_2,v_2) \tag{7}$$

describes the probability to find, at time t, a particle lying at the position $\widetilde{x}_1 \in x_1 + dx_1$ and with the velocity $\widetilde{v}_1 \in v_1 + dv_1$, as well as another particle lying at $\widetilde{x}_2 \in x_2 + dx_2$ with velocity $\widetilde{v}_2 \in v_2 + dv_2$. By taking into account the conserved quantities, and by separating the quantities which imply a particle at speed v *before* or *after* the collision (decomposition of the collision term in the *gain term* and the *loss term*), we can write:

$$\partial_t f + v\cdot\nabla_x f = \int_{\omega\in\mathbb{S}^{d-1}}\int_{v_*\in\mathbb{R}^d}\Big[B(\omega,v',v_*')f^{(2)}(t,x,v',x,v_*') - B(\omega,v,v_*)f^{(2)}(t,x,v,x,v_*)\Big]\,d\omega dv_*, \tag{8}$$

where $B(\omega,v,v_*)$ is the *collision kernel*, which is the density of probability describing how likely a collision, involving two particles with respective pre-collisional velocities v and v_*, colliding with impact parameter ω, can take place. In the case of the hard spheres (as well as for other models), we see that

$$B(\omega,v,v_*) = B(\omega,v',v_*'), \tag{9}$$

because in the present case we have

$$B(\omega,v,v_*) = |\omega\cdot(v_*-v)|. \tag{10}$$

This property is called *microreversibility*. Furthermore, by Galilean invariance of the laws of evolution of the particles, we find:

$$B(\omega,v,v_*) = B(|v_*-v|,\cos\theta), \tag{11}$$

where

$$\cos\theta = \omega\cdot\frac{v_*-v}{|v_*-v|}\cdot \tag{12}$$

Finally, if we assume that two colliding particles are uncorrelated (that is, if we assume that they did not collide with each other in the past, and that we cannot connect them with a series of collisions which occurred in the past involving other particles of the system), then we have the tensorization of the 2-particle density :

$$f^{(2)}(t,x,v,x,v_*) = f(t,x,v)f(t,x,v_*) \quad \text{et} \quad f^{(2)}(t,x,v',x,v_*') = f(t,x,v')f(t,x,v_*'). \tag{13}$$

This hypothesis is called molecular chaos, or *Stoßzahlansatz*.
We obtain then the Boltzmann equation (1872, [7]) in its general form:

$$\partial_t f + v\cdot\nabla_x f = Q(f,f) =: \int_{\omega\in\mathbb{S}^{d-1}}\int_{v_*\in\mathbb{R}^d} B(v-v_*,\omega)\left[f'f_*' - ff_*\right]d\omega dv_*, \tag{14}$$

with $f := f(t,x,v)$, $f_* := f(t,x,v_*)$, $f' := f(t,x,v')$ et $f_*' := f(t,x,v_*')$.

1.2 Properties of the Boltzmann equation

The Boltzmann equation (14) has proven to be an extremely powerful tool. Furthermore, if we define entropy as the functional:

$$H(f)(t) := \int_{x,v} \ln\left(f(t,x,v)\right) f(t,x,v) dx dv, \tag{15}$$

one can show that

$$\frac{d}{dt} H(f)(t) \leq 0, \tag{16}$$

with equality if and only if f is an equilibrium of the Boltzmann equation, that is to say a stationary solution, which must then necessarily have the form:

$$f(t,x,v) = \rho(t,x) \exp\left(-a(t,x)|v - u(t,x)|^2\right), \tag{17}$$

where $\rho, a : [0,T] \times \mathbb{R}^d \to \mathbb{R}_+$ et $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. A function of the form (17) is called a *local Maxwellian.*

The inequality (16) is the key result of Boltzmann's *H-theorem* (1872, [7]). This result implies that the solutions of the Boltzmann equation evolve irreversibly, tending to minimize the (mathematical) entropy (15). We obtain a description of the asymptotic behavior of the solutions of the Boltzmann equation: we expect these solutions to converge towards the equilibria, and moreover in certain cases of boundary domains, under certain boundary conditions, we can show that only Maxwellians of the form

$$m(t,x,v) = \rho_0 \exp\left(-a_0|v|^2\right) \tag{18}$$

(that is, with ρ and a constant, and u equal to zero in (17)) are solutions of the Boltzmann equation. Such a function, of the form (18), is called a *global Maxwellian.* We can therefore conjecture that the solutions of the Boltzmann equation converge towards the unique global Maxwellian associated with each of these solutions.

This irreversible behavior, the first of its kind to have been identified among the kinetic equations, corresponds to the evolution of fluids that we can observe on a daily basis at our scale: there exists an *arrow of time*, the fluids having tendency to evolve in such a way that they minimize their entropy.

The Boltzmann equation is an extremely relevant object for modeling highly diluted gases. In particular, we can cite the application of the Boltzmann equation to aerospace: the atmospheric reentry trajectories of spacecrafts (such as the space shuttle) are calculated using the Boltzmann equation. Indeed, the upper layers of the atmosphere constitute an example of highly diluted gas.

But the Boltzmann equation is relevant not only in the context of dilute gases, since it has also close links with fluid mechanics. Indeed, in the case of a solution of the Boltzmann equation close to a Maxwellian (17), one can develop the functions ρ, a and u in series and study the perturbations of these functions. We then find, depending on the hypotheses, that these perturbations verify the Euler equation, compressible or incompressible, or the Navier-Stokes system. The study of these connections is called *hydrodynamic limits.* ([25], [45]).

1.3 Loschmidt's and Zermelo's paradoxes, discussion of the model

As we have seen, the properties of the Boltzmann equation are rich and deep. In particular, the existence of the arrow of time for solutions of the Boltzmann equation has serious consequences. However, this result raises questions. Indeed, if we reconsider the formal derivation, presented in Section 1.1, we see that our starting point was the system of hard spheres. However, the system of hard spheres is a system subject to Newton's laws only, which is a *conservative* framework. In

particular, these systems are *time reversible*, that is to say, given a trajectory of a system of hard spheres given over a time interval $[0, T]$, considering the evolution of this system backwards, from time T, still produces a trajectory which verifies Newton's equations. There is no arrow of time in this case. As for the Boltzmann equation, we have seen that the entropy (15) is strictly decreasing during the evolution of the solutions. It is then impossible to reverse the direction of time and find another solution.
So, how is it possible that a system, reversible at the microscopic scale, can lead to obtain a system at the macroscopic scale, which is itself irreversible? This questioning led Loschmidt [35], then Zermelo [55] to formulate paradoxes which a priori undermine the validity of Boltzmann's equation. We will in fact see that these paradoxes are only apparent.

1.4 Granular gases : kinetic theory for dissipative particle systems

A model for inelastic collisions. The theory of the Boltzmann equation concerns gases whose elementary components have conservative dynamics: the kinetic energy is conserved at each collision, and we recover this property at the macroscopic scale for the solutions of the Boltzmann equation. We can also consider systems composed of particles which interact via inelastic collisions, where a fraction of the kinetic energy of the particles which collide is lost (transformation into heat, deformation of the particles, emission of photons, etc.). For example, we can assume that the particles evolve according to the model of *inelastic hard spheres*. As for the elastic case, we assume that the particles have inertial motion when they are at a sufficient distance from each other, and we modify the collision law (3) in the following way:

$$\left\{ \begin{array}{rcl} v' & = & v - \dfrac{(1+r)}{2}\left((v-v_*)\cdot\omega\right)\omega, \\ v'_* & = & v_* + \dfrac{(1+r)}{2}\left((v-v_*)\cdot\omega\right)\omega. \end{array} \right. \tag{19}$$

$r \in [0, 1]$ is called the *restitution coefficient*, it quantifies the degree of inelasticity of collisions between particles. For $r = 1$, we recover the elastic case (3), and for $r = 0$, we have the *perfectly inelastic* case. In particular, in dimension $d = 1$, when $r = 0$ and two particles collide, the particles dissipate completely their relative velocity, so that they remain attached after a collision. For this reason, the case $r = 0$ is sometimes called the case of *sticky particles*.

A model with varied and surprising applications. These models have a gigantic field of application, since sand, wheat in a silo, snow, or on other scales interstellar dust or the components of Saturn's rings can be described with the help of an inelastic particle model.
Systems composed of a large number of inelastic particles are called *granular gases*, for a reason that we will develop below. These systems exhibit fascinating behaviors, halfway between solids and liquids ([32]), and they are still largely misunderstood.

From particle systems to kinetic equations. In the same way that we formally obtained the Boltzmann equation from a microscopic-scale description of matter, we can write a kinetic equation to describe systems composed of a very large number of inelastic particles. We then obtain the *inelastic Boltzmann equation*. We can consult for example [9], and [12] for a more mathematical approach to the subject.
One of the typical phenomena observed when studying granular gases using kinetic equations is the so-called *homogeneous cooling*. Unlike the elastic case, the particles dissipate kinetic energy, and the gas loses temperature (*Haff's law* [28], [9]).
Then comes the moment when the gas has lost too much energy, and is no longer able to fill the space homogeneously: we then observe the spontaneous formation of inhomogeneity in the gas. The particles

tend to concentrate in smaller and smaller regions, in increasingly dense clouds, almost completely deserting certain parts of the previously occupied space ([9], [43]). This tendency to form agglomerates gave its name to the fluids we describe: *granular gases*.

The question of the derivation, the problem of inelastic collapse. While we know a proof of the rigorous derivation of the classical Boltzmann equation (described in detail in Section 2), at least for short times and in certain particular situations, the question of the rigorous derivation of the kinetic equations of granular gases is still largely open.
In the elastic case, a preliminary result to the rigorous derivation of the Boltzmann equation consists in proving that the system of hard spheres has a well-defined dynamics. More precisely, it is possible to show that this dynamics is globally well-defined, for *almost every* initial configuration: this is Alexander's theorem ([1], [24]).
In the inelastic case, to our knowledge no such result has been obtained. In fact, there is a phenomenon, typical of the collision model (19), called the *inelastic collapse*, which can occur even with a system composed of a very small number of particles. A system of particles undergoes inelastic collapse when an infinite number of collisions occur in a finite time. Such a phenomenon, first observed numerically, then rigorously studied in dimension 1 ([46], [5], [38], [17], [27], [16],[3], and more recently [15] and [31]), was subsequently observed in dimension 2 [39], then studied [56], but remains still largely misunderstood. In particular, even the case of a system of four particles in dimension 1 has not yet revealed all its mysteries ([16], [31]). It should be noted that the case of the dimension 1 is not anecdotal, since, when an inelastic collapse takes place in any dimension, the particles involved in the collisions seem to arrange themselves into structures which have the form of chains, rather flat and almost linear ([39], [43]).
The phenomenon of inelastic collapse is therefore a major obstruction to obtain an Alexander theorem for inelastic particles. Such a theorem concerning the well-posed character of the particle dynamics of a granular gas constitutes itself a necessary and fundamental step in order to obtain a rigorous derivation of the inelastic kinetic equations.

2 Rigorous derivation, Boltzmann-Grad limit and Lanford's theorem

We will see in this part how to rigorously derive the Boltzmann equation from a system of elastic hard spheres. The idea is to represent the gas, which is a continuum, by a system composed of a very large number of particles, while maintaining a very low density.

2.1 BBGKY hierarchy and Boltzmann-Grad limit

The starting point consists of describing precisely the evolution of a system of hard spheres, with the appropriate tools from statistical physics. In particular, we will use the distribution function of the system, and study its marginals. For more details, please the reader may refer to [13], [14] et [24].

The BBGKY hierarchy. For a system of N hard spheres of radius $\varepsilon/2 > 0$, we start with defining the *configuration* of the system: this is the vector Z_N in which the positions and velocities of each particles are collected. We have:

$$Z_N = (x_1, v_1, \dots, x_N, v_N) \in \mathbb{R}^{2dN}, \tag{20}$$

where x_i and v_i represent respectively the position and velocity of the particle i. Since the particles cannot overlap, the vector Z_N belongs to the *phase space* $\mathcal{D}^\varepsilon_N$, defined as:

$$\mathcal{D}^\varepsilon_N = \Big\{ Z_N \in \mathbb{R}^{2dN} \,/\, \forall\, i \neq j,\, |x_i - x_j| > \varepsilon \Big\}. \tag{21}$$

We now define the distribution function f_N of the system: $f_N(t,Z_N)$ represents the probability of finding, at time t, the first particle in x_1+dx_1, moving at speed $w_1 \in v_1+dv_1$, the second in x_2+dx_2 and moving at speed $w_2 \in v_2+dv_2$, and so on.
By definition, particles have inertial motion within the phase space $\mathcal{D}_N^\varepsilon$, so that:

$$\forall Z_N \in \mathcal{D}_N^\varepsilon, \qquad \partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0. \tag{22}$$

The equation (22) is called the *Liouville equation*, that we complete with the following boundary conditions:

$$\begin{aligned} f_N(t,x_1,v_1,\dots,x_i,v_i,\dots,x_j,v_j,\dots,x_N,v_N)& \\ = f_N(t,x_1,v_1,\dots,x_i,v_i',\dots,x_j,v_j',\dots,x_N,v_N)& \end{aligned} \tag{23}$$

when $|x_i - x_j| = \varepsilon$, with v_i' and v_j' denoting the post-collisional velocities computed for the pair i, j from the pre-collisional velocities v_i, v_j, given by the relation (3).
Now, we will focus on the typical behavior of a subgroup of s particles of the system. Such information is encoded in the *s-th marginal $f_N^{(s)}$ of the distribution function* f_N, defined as:

$$f_N^{(s)}(t,Z_s) = \int_{\mathbb{R}^{2d(N-s)}} f_N(t,Z_s,x_{s+1},v_{s+1},\dots,x_N,v_N)\mathbb{1}_{\mathcal{D}_N^\varepsilon} dx_{s+1}dv_{s+1}\dots dx_N dv_N. \tag{24}$$

In the same way that the distribution function f_N satisfies the Liouville equation in the phase space with N particles $\mathcal{D}_N^\varepsilon$, the s-th marginal $f_N^{(s)}$ (with $1 \leq s \leq N-1$) verifies the equation :

$$\partial_t f_N^{(s)} + \sum_{i=1}^s v_i \cdot \nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1}^{N,\varepsilon} f_N^{(s+1)} \tag{25}$$

in the phase space with s particles $\mathcal{D}_s^\varepsilon$, where the term $\mathcal{C}_{s,s+1}^{N,\varepsilon} f^{(s+1)}$, which involves the $(s+1)$-th marginal $f_N^{(s+1)}$, is defined as:

$$\mathcal{C}_{s,s+1}^{N,\varepsilon} f^{(s+1)} = \sum_{i=1}^s (N-s)\varepsilon^{d-1} \int_{\mathbb{S}_\omega^{d-1}} \int_{\mathbb{R}_{v_{s+1}}^d} \omega \cdot (v_{s+1} - v_i) f_N^{(s+1)}(t,Z_s,x_i+\varepsilon\omega,v_{s+1}) d\omega dv_{s+1}. \tag{26}$$

The family of equations (25), (26) for $1 \leq s \leq N-1$, completed by the Liouville equation (22), constitutes the *BBGKY hierarchy* ([6], [8], [33], [54]). It describes using the evolution of the particle system with the help of a PDE.

The Boltzmann-Grad limit. We consider the limit $N \to +\infty$ for the equations (25), (26) of the BBGKY hierarchy. Grad was the first to observe that, to obtain a meaningful limit, one must have:

$$N\varepsilon^{d-1} \xrightarrow[N\to+\infty]{} 1. \tag{27}$$

This is the *Boltzmann-Grad limit* ([26], [13], [14], [24]). The volume occupied by the particles tends to 0 as N tends to infinity. For this reason, the Boltzmann-Grad limit is also called sometimes the *low density limit*. We naturally find the hypothesis of strong dilution of gases which we seek to describe with the Boltzmann equation.
Finally, for the first marginal $f_N^{(1)}$, we see that in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, and in the case where the second marginal $f^{(2)}$ is tensorized, the first marginal of the BBGKY hierarchy is a (formal) solution of the Boltzmann equation.
We therefore have a clear plan for the rigorous derivation of the Boltzmann equation: we must study the convergence of the first marginal of the BBGKY hierarchy in the Boltzmann-Grad limit.

The Boltzmann hierarchy and the plan for the rigorous derivation. Let us proceed to a final preparatory step before carrying out the rigorous derivation of the Boltzmann equation. We first introduce the formal limit of the BBGKY hierarchy, as follows. We consider the following sequence of equations (infinite, this time), for all $s \in \mathbb{N}^*$:

$$\partial_t f^{(s)}(t, Z_s) + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i} f^{(s)}(t, Z_s) = \mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) \tag{28}$$

with

$$\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) = \sum_{i=1}^{s} \int_{\mathbb{S}^{d-1}_\omega} \int_{\mathbb{R}^d_{v_{s+1}}} [\omega \cdot (v_{s+1} - v_i)]_+ \Big[f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i', \dots, x_i, v_{s+1}') \\ - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big] d\omega dv_{s+1}. \tag{29}$$

This infinite hierarchy is called the *Boltzmann hierarchy.*

The rigorous study of the Boltzmann-Grad limit of the BBGKY hierarchy is then carried out in two steps following Lanford's approach ([34], [13], [14], later revisited and completed in [24], and detailed in [19]).

2.2 Rigorous derivation I: solving the hierarchies.

Rewriting the hierarchies as a fixed point problem. First, we transform the two hierarchies (25), (26) and (28), (29) into two similar fixed point problems. We rewrite (formally, for the moment) the hierarchies in their time-integrated form, which provides (in the case of the Boltzmann hierarchy):

$$f^{(s)} = \mathcal{T}^{s,0}_t f^{(s)}_0 + \int_0^t \mathcal{T}^{s,0}_{t-u} \mathcal{C}^0_{s,s+1} f^{(s+1)} du, \tag{30}$$

where $\mathcal{T}^{s,0}_t$ is the backwards transport operator (hard sphere transport in the case of the BBGKY hierarchy, and free transport for the Boltzmann hierarchy), defined as follows:

$$\left(\mathcal{T}^{s,0}_t f^{(s)} \right)(t, Z_s) = f\left(t, T^{s,0}_{-t}(Z_s) \right). \tag{31}$$

We denoted by $T^{s,0}_t(Z_s)$ the image of a configuration Z_s by the free transport, transported for a time t. Similarly, we will denote by $T^{s,\varepsilon}_t(Z_s)$ the image of the configuration by the transport of hard spheres of radius $\varepsilon/2$.
We now introduce the following functional spaces in order to solve the equations (30). We will present here only the spaces concerning the Boltzmann hierarchy. The spaces concerning the BBGKY hierarchy are defined in a similar way, but present difficulties which will be discussed below (see Section 3.1, which presents the article [20], where these spaces are discussed in a precise way).

Definition 2.1 (Norm $|\cdot|_{s,\beta}$, spaces $X_{s,\beta}$)**.** Let $\beta > 0$ be a strictly positive number and s be a strictly positive integer. For any measurable function $f^{(s)} : \mathbb{R}^{2ds} \to \mathbb{R}$, we define the norm:

$$|f^{(s)}|_{s,\beta} = \sup_{Z_s \in \mathbb{R}^{2ds}} \text{ess} \left[\left| f^{(s)}(Z_s) \right| \exp\left(\frac{\beta}{2} \sum_{i=1}^{s} |v_i|^2 \right) \right], \tag{32}$$

and we introduce the spaces $X_{s,\beta}$ as the set of measurable functions $f^{(s)} : \mathbb{R}^{2ds} \to \mathbb{R}$ with a norm $|\cdot|_{s,\beta}$ finite, that is:

$$X_{s,\beta} = \Big\{ f^{(s)} : \mathbb{R}^{2ds} \to \mathbb{R} \text{ measurable } / \ |f^{(s)}|_{\varepsilon,s,\beta} < +\infty \Big\}. \tag{33}$$

These first spaces will be used to control the decay at infinity of the marginals. Each marginal $f^{(s)}$ will belong to a space $X_{s,\beta}$. We mentioned that we solve hierarchies by seeing them as fixed point problems. However, the equations of the hierarchies are not closed: the equation (28) involves the s-th and $(s+1)$-th marginals. We will therefore consider the marginals together, grouped into a single vector. In other words, we will not try to solve the equations (30) independently for all s, but we will rather consider sequences $\big(f^{(s)}\big)_s$, which must belong to a Banach space to be defined, so that each coordinate $f^{(s)}$ is a solution of the s-th equation (30). It is then appropriate to "link" these marginals, by defining a norm on these vector spaces, each component of which is a marginal.

Definition 2.2 (Norm $\|\cdot\|_{\beta,\mu}$, space $\mathbf{X}_{\beta,\mu}$)**.** Let $\beta > 0$ be a strictly positive number and μ be a real number. For any sequence $F = \big(f^{(s)}\big)_{s\geq 1}$ of functions $f^{(s)}$ of $X_{s,\beta}$, we define the norm:

$$\|F\|_{\beta,\mu} = \sup_{s\geq 1} \Big(|f^{(s)}|_{s,\beta} \exp(s\mu) \Big), \tag{34}$$

and the space $\mathbf{X}_{\beta,\mu}$ as the set of sequences of functions $F = \big(f^{(s)}\big)_{s\geq 1}$ such that for all $s \geq 1$, $f^{(s)}$ belongs to the space $X_{s,\beta}$, and such that the sequence $\big(h_N^{(s)}\big)_{1\leq s\leq N}$ has a finite $\|\cdot\|_{\beta,\mu}$ norm, that is:

$$\mathbf{X}_{\beta,\mu} = \Big\{ F = \big(f^{(s)}\big)_{s\geq 1} \in \big(X_{s,\beta}\big)_{s\geq 1} \ / \ \|F\|_{\beta,\mu} < +\infty \Big\}.$$

Finding a stable functional space under the action of the collision operator. One of the major difficulties that arise when trying to solve hierarchies as a fixed point problem is to find an invariant space under the action of the collision term. In particular, if we consider the expression (29), we see that if we assume that the $(s+1)$-th marginal $f^{(s+1)}$ is bounded by a Gaussian profile of the form:

$$|f^{(s+1)}(Z_{s+1})| \leq C \exp\left(-\frac{\beta}{2} \sum_{i=1}^{s+1} |v_i|^2 \right), \tag{35}$$

it is not possible a priori to deduce such a bound for the s-th marginal $f^{(s)}$, since the collision kernel $\omega \cdot (v_{s+1} - v_i)$ induces a loss of decay at infinity. We can then proceed in two different ways:

- assign a different Gaussian weight $\beta = \beta_s$ for each marginal, but then we face the fact that we cannot consider the marginals obtained as tensorizations of the first marginal (which excludes the case of equilibrium, and is therefore not relevant here),
- or consider a Gaussian weight $\beta = \beta(t)$, uniform in s, but decreasing in time. This approach is justified by the fact that hierarchies rewritten as the fixed point problem (30) have been integrated in time, so that taking advantage of this additional time variable, and with the help of appropriate weights, we can hope to obtain invariant spaces under the action of the collision operator, integrated in time.

These points are discussed in [24], and taken up in detail and clarified in [19], and [20].
It is therefore necessary to consider more sophisticated spaces and norms. This is the purpose of the following definition.

Definition 2.3 (Norm $|\!|\!|\cdot|\!|\!|_{\tilde{\beta},\tilde{\mu}^1}$, space $\widetilde{\mathbf{X}}_{\tilde{\beta},\tilde{\mu}}$)**.** For any strictly positive real $T > 0$, any strictly positive and strictly decreasing function $\tilde{\beta}$, any strictly decreasing function $\tilde{\mu}$, both defined on $[0,T]$, and any function $\widetilde{F} : [0,T] \to \bigcup_{t\in[0,T]} \mathbf{X}_{\tilde{\beta}(t),\tilde{\mu}(t)}$, $t \mapsto \widetilde{F}(t) = \big(f^{(s)}(t)\big)_{s\geq 1}$ such that $\widetilde{F}(t)\ in \mathbf{X}_{\tilde{\beta}(t),\tilde{\mu}(t)}$ for all $t \in [0,T]$, we define

$$\left|\!\left|\!\left| \widetilde{F} \right|\!\right|\!\right|_{\tilde{\beta},\tilde{\mu}} = \sup_{0\leq t\leq T} \left\| \widetilde{F}(t) \right\|_{\tilde{\beta}(t),\tilde{\mu}(t)},$$

and we define the space $\widetilde{\mathbf{X}}_{\widetilde{\beta},\widetilde{\mu}}$ as the set of such functions $\widetilde{F}$ with a norm $|||\cdot|||_{\widetilde{\beta},\widetilde{\mu}}$ that is finite, and left-continuous in time in the following sense:

$$\forall t \in]0,T],\ \forall\ s \geq 1, \quad \lim_{u \to t^-} \big| f^{(s)}(t) - f^{(s)}(u) \big|_{s,\widetilde{\beta}(t)} = 0. \tag{36}$$

The spaces introduced above are Banach spaces.
We obtain then the first important result in order to derive the Boltzmann equation.

Theorem 2.4 (Nishida 1977 [41], Ukai 2000 [52], Gallagher, Saint-Raymond, Texier 2014 [24])**.** *Let β_0 be a strictly positive number, and μ_0 be a real number.*
Then there exists a time $T = T(\beta_0,\mu_0) > 0$, a strictly positive and strictly decreasing function $\widetilde{\beta}$: $[0,T] \to \mathbb{R}_+^*$ and a strictly decreasing function $\widetilde{\mu} : [0,T] \to \mathbb{R}$ such that

$$\widetilde{\beta}(0) = \beta_0, \qquad \widetilde{\mu}(0) = \mu_0, \tag{37}$$

and such that for any integer N, in the Boltzmann-Grad limit $N\varepsilon^{d-1} = 1$, there exists for any pair of initial data $F_{N,0} \in \boldsymbol{X}_{N,\varepsilon,\beta_0,\mu_0}$ and $F_0 \in \boldsymbol{X}_{0,\beta_0,\mu_0}$ a unique pair of solutions, to the BBGKY hierarchy, respectively to the Boltzmann hierarchy, in the spaces $\widetilde{\boldsymbol{X}}_{N,\varepsilon,\widetilde{\beta},\widetilde{\mu}}$, respectively $\widetilde{\boldsymbol{X}}_{0,\widetilde{\beta},\widetilde{\mu}}$.

Let us note that the existence time given in the theorem is explicit, but depends directly on the size of the initial data. This phenomenon is appears also in the methods for solving the Boltzmann equation with fixed point methods, in the absence of the possibility of taking into account the cancellations which take place between the loss term and the gain term of the collision term of the Boltzmann equation.
Furthermore, this time is typically extremely small. We can only carry out the rigorous derivation of the Boltzmann equation on this time interval $[0,T]$.

2.3 Rigorous derivation II: convergence of the BBGKY hierarchy

Now that we found solutions of the BBGKY and Boltzmann hierarchies, over the same time interval $[0,T]$, and this, uniformly in the number of particles N of the system of hard spheres, we will be able to study the sequence of solutions $(F_N)_{N\geq 1}$ of the BBGKY hierarchy, in the Boltzmann-Grad limit.

From the explicit formula to the decomposition in elementary terms. In our case, the result of Theorem 2.4 is based on Banach's fixed point theorem. This theorem is constructive, and we can obtain an explicit expression of the solutions of the hierarchies, with the *Duhamel formulas*. The solutions F_N and F are then described only in terms of the initial data $F_{N,0}$ and F_0. More precisely, if we introduce the notations:

$$\mathcal{I}_s^{N,\varepsilon} f_N^{(s+1)} = \int_0^t \mathcal{T}_{-u}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_u^{s+1,\varepsilon} f_N^{(s+1)}(u,\cdot)\,\mathrm{d}u \tag{38}$$

for the BBGKY hierarchy, and

$$\mathcal{I}_s^0 f^{(s+1)} = \int_0^t \mathcal{T}_{-u}^{s,0} \mathcal{C}_{s,s+1}^0 \mathcal{T}_u^{s+1,0} f^{(s+1)}(u,\cdot)\,\mathrm{d}u \tag{39}$$

for the Boltzmann hierarchy, then the composition of such integrated in time transport-collision operators, defined according to the formulae:

$$\mathcal{I}_{s,s+k-1}^{N,\varepsilon} f_N^{(s+k)} = \int_0^t \mathcal{T}_{-t_1}^{s,\varepsilon} \mathcal{C}_{s,s+1}^{N,\varepsilon} \mathcal{T}_{t_1}^{s+1,\varepsilon} \int_0^{t_1} \mathcal{T}_{-t_2}^{s+1,\varepsilon} \mathcal{C}_{s+1,s+2}^{N,\varepsilon} \mathcal{T}_{t_2}^{s+2,\varepsilon} \dots \int_0^{t_{k-1}} \mathcal{T}_{-t_k}^{s+k-1,\varepsilon} \mathcal{C}_{s+k-1,s+k}^{N,\varepsilon} \mathcal{T}_{t_k}^{s+k,\varepsilon} f_N^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \dots \mathrm{d}t_2\,\mathrm{d}t_1 \tag{40}$$

and

$$\mathcal{I}^0_{s,s+k-1} f^{(s+k)} = \int_0^t \mathcal{T}^{s,0}_{-t_1} \mathcal{C}^0_{s,s+1} \mathcal{T}^{s+1,0}_{t_1} \int_0^{t_1} \mathcal{T}^{s+1,0}_{-t_2} \mathcal{C}^0_{s+1,s+2} \mathcal{T}^{s+2,0}_{t_2} \dots$$
$$\int_0^{t_{k-1}} \mathcal{T}^{s+k-1,0}_{-t_k} \mathcal{C}^0_{s+k-1,s+k} \mathcal{T}^{s+k,0}_{t_k} f^{(s+k)}(t_k,\cdot)\,\mathrm{d}t_k \dots \mathrm{d}t_2\,\mathrm{d}t_1 \tag{41}$$

respectively for the BBGKY and Boltzmann hierarchies, we can then rewrite the solutions of the hierarchies given by Theorem 2.4 in the following form, for example for the Boltzmann hierarchy:

$$F = t \mapsto \Big(\mathcal{T}^{s,0}_t f^{(s)}_0(\cdot) + \sum_{k=1}^{+\infty} \mathcal{I}^0_{s,s+k-1}\big(u \mapsto \mathcal{T}^{s+k,0}_u f^{(s+k)}_0\big)(t,\cdot)\Big)_{s\geq 1}. \tag{42}$$

Interpretation in terms of pseudo-trajectories. Let us now present Lanford's last argument [34], which allows us to conclude concerning the convergence of the marginals, through a geometric interpretation of the expressions (42) of the solutions of the hierarchies.
First, let us observe that each integrated transport-collision operator $\mathcal{I}^{\cdot}_{s,s+1}$ is naturally decomposed as follows: the collision operators $\mathcal{C}^{\cdot}_{s,s+1}$ given by (26) and (29) are sums of s integrals, each of which decomposes into a gain term (sign $+$) and a loss term ($-$). We are therefore naturally led to introduce the elementary integrated transport-collision operators:

$$\mathcal{C}^0_{s,s+1} f^{(s+1)} = \int_0^t \mathcal{T}^{s,0}_{t-t_1} \mathcal{C}^0_{s,s+1} \mathcal{T}^{s+1,0}_{t_1} f^{(s+1)}_0(t_1,\cdot)\,\mathrm{d}t_1 =: \sum_{j_1=1}^{s} \Big(\mathcal{I}^0{}^{\,s}_{+,j_1} - \mathcal{I}^0{}^{\,s}_{-,j_1}\Big)\big(t_1 \mapsto \mathcal{T}^{s+1,0}_{t_1} f^{(s+1)}_0\big), \tag{43}$$

which provides the following decomposition:

$$\mathcal{I}^0_{s,s+k-1} := \big(\sum_{\substack{1\leq j_1\leq s\\ \pm_1}} (\pm_1) \mathcal{I}^0{}^{\,s}_{\pm_1,j_1} \big) \circ \dots \circ \big(\sum_{\substack{1\leq j_k\leq s+k-1\\ \pm_k}} (\pm_k) \mathcal{I}^0_{\substack{s+k-1\\ \pm_k,j_k}} \big). \tag{44}$$

The solutions of the hierarchies are then rewritten (here, in the case of the Boltzmann hierarchy, but a similar decomposition also holds for the BBGKY hierarchy):

$$F = \Big(\mathcal{T}^{s,0}_t f^{(s)}_0 + \sum_{k=1}^{+\infty} \sum_{J_k,M_k} (\pm_1)\dots(\pm_k) \mathcal{I}^0_{\substack{s,s+k-1\\ J_k,M_k}}\big(u \mapsto \mathcal{T}^{s+k,0}_u f^{(s+k)}_0\big)\Big)_{s\geq 1}, \tag{45}$$

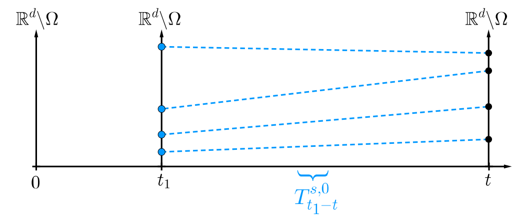
where $J_k = (j_1,\dots,j_k)$, $M_k = (\pm_1,\dots,\pm_k)$, and $s \leq j_l \leq s+l-1$. The expression (45) of the solution F is called its *decomposition in elementary terms.*
These are the elementary terms that we will be able to interpret geometrically. Let us consider an example. In the formula (45), the term obtained for arbitrary $k=1$, $m_1 = -$, and s, $j_1 = j$, the operator $\mathcal{I}^0{}_{hspace-2mm}{}^{s,s}_{(j_1,-)}\big(u \mapsto \mathcal{T}^{s+1,0}_u f^{(s+1)}_0\big)$ is written explicitly:

$$\int_0^t \int_\omega \int_{v_{s+1}} \big[\omega \cdot \big(v_{s+1} - (T^{s,0}_{t_1-t}(Z_s))^{V,j}\big)\big]_-$$
$$\times f^{(s+1)}_0\big(T^{s+1,0}_{-t_1}\big(T^{s,0}_{t_1-t}(Z_s), \big(T^{s,0}_{t_1-t}(Z_s)\big)^{X,j}, v_{s+1}\big)\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1. \tag{46}$$

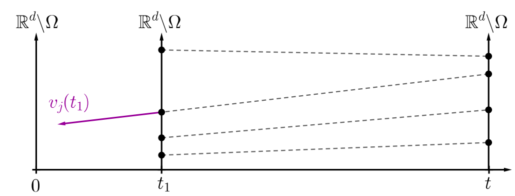
Let us now interpret the formula (46), step by step. This interpretation is presented using Figure 1. The left column contains repetitions of (46), where we highlight the respective terms which are represented geometrically in the right column. In this second column, we will represent the time axis horizontally, and for each abscissa, a copy of the space $\mathbb{R}^d$ is represented vertically (in a schematic way, since we only have one dimension available...). The position of the different particles, at a given

$$\int_0^t\int_\omega\int_{v_{s+1}} \big[\omega\cdot\big(v_{s+1} - (T^{s,0}_{t_1-t}(\overbrace{Z_s}^{\textcircled{1}}))^{V,j}\big)\big]_-$$
$$\times f_0^{(s+1)}\big(T^{s+1,0}_{-t_1}\big(T^{s,0}_{t_1-t}(\underbrace{Z_s}_{\textcircled{1}}), (T^{s,0}_{t_1-t}(\underbrace{Z_s}_{\textcircled{1}}))^{X,j}, v_{s+1}\big)\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$

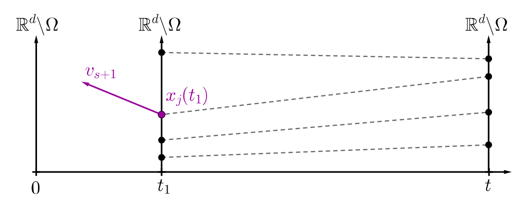
$$\int_0^t\int_\omega\int_{v_{s+1}} \big[\omega\cdot\big(v_{s+1} - \overbrace{(T^{s,0}_{t_1-t}(Z_s))}^{\textcircled{2}}{}^{V,j}\big)\big]_-$$
$$\times f_0^{(s+1)}\big(T^{s+1,0}_{-t_1}\big(\underbrace{T^{s,0}_{t_1-t}(Z_s)}_{\textcircled{2}}, \underbrace{(T^{s,0}_{t_1-t}(Z_s))}_{\textcircled{2}}{}^{X,j}, v_{s+1}\big)\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$



$$\int_0^t\int_\omega\int_{v_{s+1}} \big[\omega\cdot\big(v_{s+1} - \overbrace{(T^{s,0}_{t_1-t}(Z_s))^{V,j}}^{\textcircled{3}}\big)\big]_-$$
$$\times f_0^{(s+1)}\big(T^{s+1,0}_{-t_1}\big(T^{s,0}_{t_1-t}(Z_s), (T^{s,0}_{t_1-t}(Z_s))^{X,j}, v_{s+1}\big)\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$



$$\int_0^t\int_\omega\int_{v_{s+1}} \big[\omega\cdot\big(v_{s+1} - (T^{s,0}_{t_1-t}(Z_s))^{V,j}\big)\big]_-$$
$$\times f_0^{(s+1)}\big(T^{s+1,0}_{-t_1}\big(T^{s,0}_{t_1-t}(Z_s), \underbrace{(T^{s,0}_{t_1-t}(Z_s))^{X,j}}_{\textcircled{4}}, \underbrace{v_{s+1}}_{\textcircled{4}}\big)\big)\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$



$$\int_0^t\int_\omega\int_{v_{s+1}} \big[\omega\cdot\big(v_{s+1} - (T^{s,0}_{t_1-t}(Z_s))^{V,j}\big)\big]_-$$
$$\times f_0^{(s+1)}\underbrace{\big(T^{s+1,0}_{-t_1}\big(T^{s,0}_{t_1-t}(Z_s), (T^{s,0}_{t_1-t}(Z_s))^{X,j}, v_{s+1}\big)\big)}_{\textcircled{5}}\,\mathrm{d}\omega\,\mathrm{d}v_{s+1}\,\mathrm{d}t_1$$

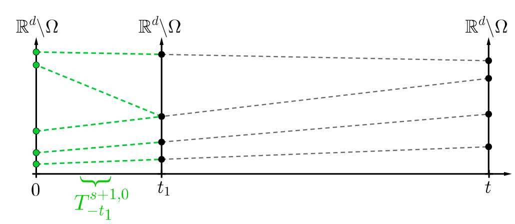


Figure 1: Construction, step by step, of a pseudo-trajectory.

time, is therefore represented by a cloud of points placed at different heights.
We start from a configuration Z_s of s particles (in red) at time t, which we transport (with backwards transport) until time t_1 (in blue). This time t_1 is the integration variable of the integral used to define the integrated transport-collision operator. In the third step, we choose the particle j, represented in burgundy at its position at time t_1, with its velocity. We add a $(s+1)$-th particle to the system at time t_1, at this same position, but with a velocity v_{s+1}, which is an integration variable of the collision operator (29). We finally transport this new system of $s+1$ particles to time 0, and we obtain the

final configuration in green.
These different configurations are the variables which appear in the expression of the elementary term (46) written explicitly.

This geometric construction, which has the shape of a tree, is called a *pseudo-trajectory*.
Lanford's observation consists in noting that, on the one hand, for two identical elementary terms, that is, associated to the same parameters k, J_k and M_k, but associated with the two hierarchies, the construction process provides very similar pseudo-trajectories.

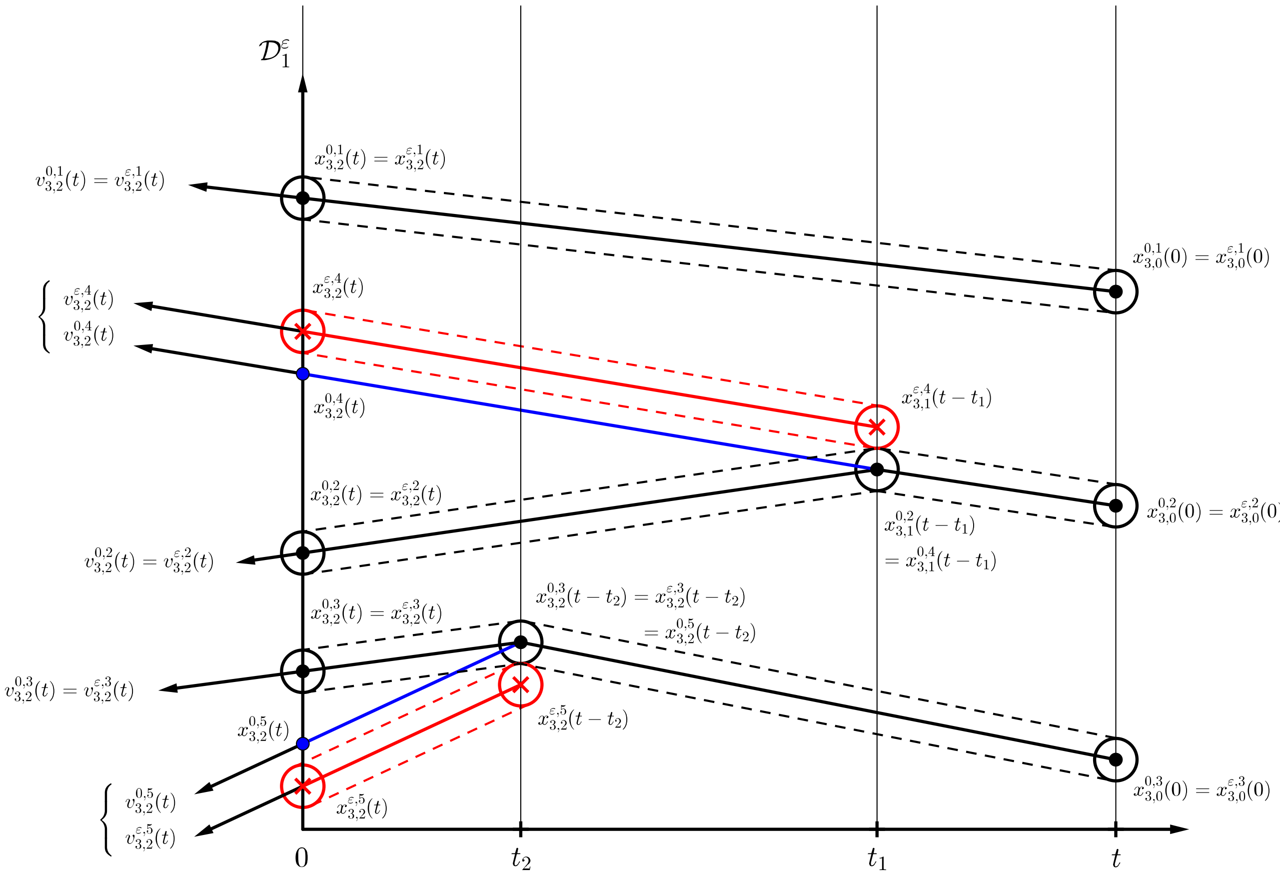


Figure 2: Comparison of the pseudo-trajectories of the two hierarchies, associated to the same elementary term.

On the other hand, these pseudo-trajectories, that is, the configurations in which we evaluate the initial distributions f_0 to define the elementary terms, are in fact the only difference between the elementary terms of the BBGKY hierarchy and of the Boltzmann hierarchy. In Figure 2, we represented the pseudo-trajectories of the two hierarchies, associated with the same elementary term. The final argument to conclude is clear: the pointwise convergence of the pseudo-trajectories will lead to the convergence of the elementary terms by the dominated convergence theorem, and the term-by-term convergence of the elements of the decomposition (45) completes the proof of the derivation. This pointwise convergence of the pseudo-trajectories is then easily obtained, since the difference between the pseudo-trajectories of the two hierarchies associated with the same elementary term is directly linked to the size ε of the hard spheres, which tends towards 0 in the Boltzmann-Grad limit.

The case of the recollisions. Nevertheless, a final obstacle appears and prevents to conclude. There is a fundamental difference between the pseudo-trajectories of the two hierarchies, which is due to the different transports used in the two cases.
In the case of the Boltzmann hierarchy, we use the free transport to write the "mild" form (30). In the case of the BBGKY hierarchy, we use the transport of hard spheres. In the first case, the particles cannot collide, while in the second case their velocities can be suddenly changed because of these collisions. Thus, two pseudo-trajectories, associated with the same elementary term, and with the same initial data and particle adjunction conditions, can be radically different between the two hierarchies. This phenomenon, illustrated in Figure 3, is called a *recollision.*
Controlling recollisions, which is a geometric problem, is a step, certainly technical, but very important to conclude. In the article [24], Gallagher, Saint-Raymond and Texier show that only an error term, small in the limit (and which corresponds to a subset of the domain of integration in the integrals which define the integrated in time transport-collision operators (26), (29), (38), (39)), is associated with pseudo-trajectories which present recollisions. They then obtain the following result, which constitutes a rigorous derivation of the Boltzmann equation (for the hard sphere collision kernel, but also for interactions via short-range, vanishing potentials in the same article) in the Boltzmann-Grad limit , in the whole Euclidean space $\mathbb{R}^d$ or in the torus $\mathbb{T}^d$. As a consequence of the explicit control of recollisions, they also obtain an explicit convergence rate of the first marginal of a system of N hard spheres towards the associated solution of the Boltzmann equation.

Theorem 2.5 (Lanford 1975 [34], Gallagher, Saint-Raymond, Texier 2014 [24])**.** *Let*

$$f_0 : \mathbb{R}^{2d} \to \mathbb{R}_+ \tag{47}$$

be a continuous probability density such that:

$$\left\| f_0(x,v) \exp\left(\frac{\beta}{2}|v|^2\right) \right\|_{L^\infty(\mathbb{R}^{2d})} < +\infty \tag{48}$$

for a certain number $\beta > 0$.
Then, in the Boltzmann-Grad limit $N \to +\infty$, $N\varepsilon^{d-1} = 1$, $f_N^{(1)}$ converges to the solution f of the Boltzmann equation with collision kernel $b(v,\omega) = (v\cdot\omega)_+$, with f_0 as initial data, in the following sense. For any compact set $K \subset \mathbb{R}^d$, and for any sufficiently regular test function $\varphi : \mathbb{R}^d \to \mathbb{R}$:

$$\left\| \mathbb{1}_K(x) \int_{\mathbb{R}^d_v} \varphi(v)\big(f_N^{(1)} - f\big)(x,v)\,\mathrm{d}v \right\|_{L^\infty([0,T]\times\mathbb{R}^d_x)} \underset{N\to+\infty}{\longrightarrow} 0.$$

If in addition f_0 is a Lipschitz function, the convergence rate is at most of order $O(\varepsilon^a)$, with

$$a < \frac{d-1}{d+1}\cdot$$

3 Contributions to the problem of the derivation of the Boltzmann equation

It is on the problem of the rigorous derivation of the Boltzmann equation, described in detail in Section 2, that I began my research, first during the thesis, then during my first post-doctorate under the supervision of Chiara Saffirio (University of Basel).

3.1 Question of the appropriate functional spaces to solve the hierarchies

The proof of Lanford's theorem (Theorem 2.5) is extremely long and complex, and it has been completed over time by many authors. A first question consists of determining in which spaces we can

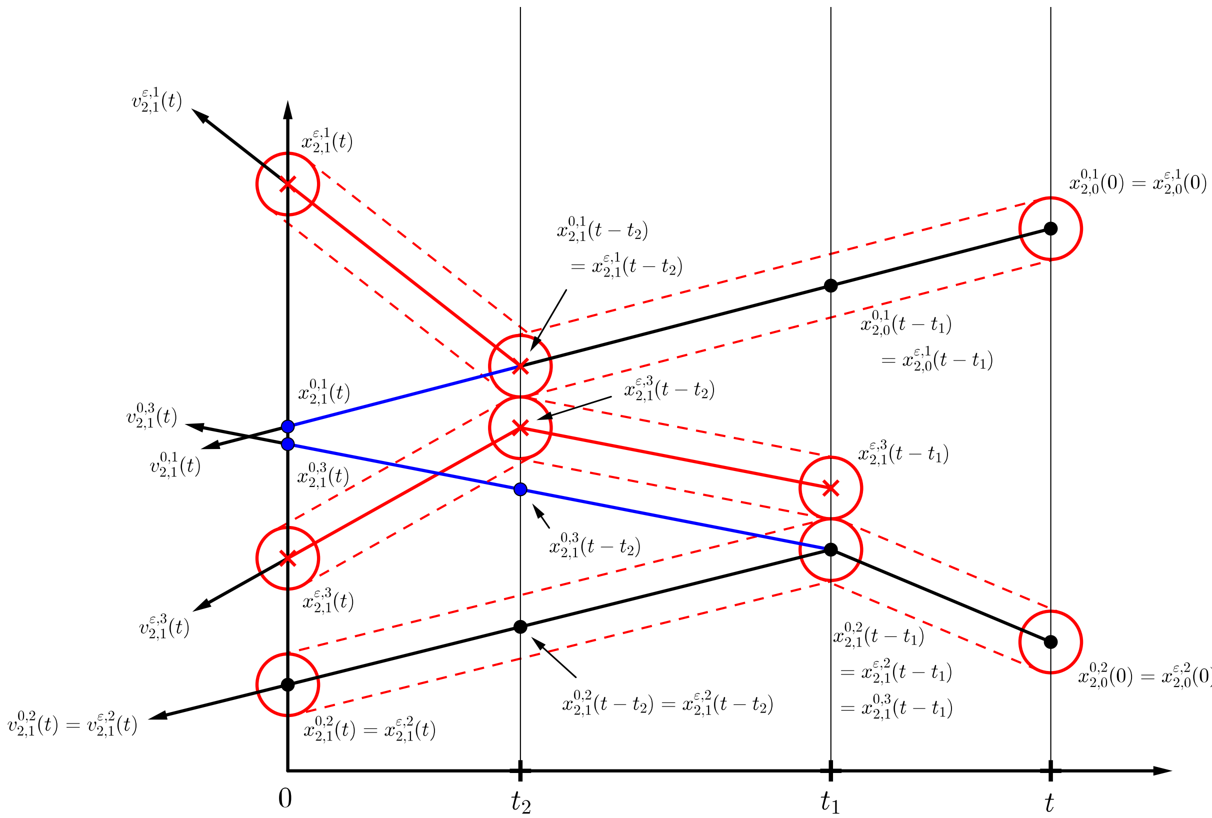


Figure 3: Comparison of pseudo-trajectories of the two hierarchies associated to the same elementary term, in the pathological case of a recollision.

resolve the hierarchies (25) and (28). It is therefore a question of functional analysis, and which is fundamental to obtain the existence of the objects used in the rigorous derivation of the Boltzmann equation.

In the case of the Boltzmann hierarchy, we can simply work in spaces of continuous functions, since the free transport preserves continuity. But in the case of the BBGKY hierarchy, which describes the evolution of the system of hard spheres, we are confronted with continuity problems, due to the nature of the particle dynamics.
In particular, due to problems coming from the definition of the hard sphere dynamics, it is not possible to consider the flow induced by the dynamics of these particles for any initial data (consider for example the case of a collision involving three particles at the same time : it is not possible to compute the post-collision velocities with the formulas (3) in this case). In particular, we can no longer work with spaces of continuous functions, and we must then use Lebesgue spaces (typically, we will consider L^∞ functions on the phase space).
But then a new problem arises, since in this case, how to define the collision operator (26) for a function $f_N^{(s+1)} \in L^\infty\left(\mathcal{D}_N^\varepsilon\right)$? Indeed, this operator is defined using an integral over a submanifold of strictly positive codimension, which does not make sense for a function L^∞ in general. We must therefore look for a trace result for the solutions of the BBGKY hierarchy. This problem gave rise to a long development in the thesis [19], in which the problem is solved in detail following a construction

proposed in [24], and synthesized in the article [20]. We obtain in particular the following result, which describes the sufficient regularity of a function $f_N^{(s+1)} \in L^\infty\left(\mathcal{D}_N^\varepsilon\right)$ for which the integrated transport-collision operator is well-defined.

Theorem 3.1 (Definition of the integrated transport-collision operator of the BBGKY hierarchy, D. 2019 [19], [20])**.** *Let s be a strictly positive integer, ε and T be two positive numbers. Let $g_{s+1} : [0,T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function such that:*

- *$(t,x) \mapsto g_{s+1}(t,x)$ is measurable and positive almost everywhere,*
- *for all $x \in \mathbb{R}_+$, the function $t \mapsto g_{s+1}(t,x)$ is increasing,*
- *for all $t \in [0,T]$ and almost all $(v_1,\dots,v_s) \in \mathbb{R}^{ds}$, the function $v_{s+1} \mapsto \big|V_{s+1}\big| g_{s+1}\big(t,\big|V_{s+1}\big|\big)$ is integrable on mathbbR^d,*
- *for all $t \in [0,T]$, the function $(v_1,\dots,v_s) \mapsto \displaystyle\int_{\mathbb{R}^d} \big|V_{s+1}\big| g_{s+1}\big(t,\big|V_{s+1}\big|\big)\,\mathrm{d}v_{s+1}$ is bounded almost everywhere, and*

$$\left\| \int_{\mathbb{R}^d} \mathbb{1}_{|V_{s+1}|\geq R} \big|V_{s+1}\big| g_{s+1}\big(t,\big|V_{s+1}\big|\big)\,\mathrm{d}v_{s+1} \right\|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))}$$

  tends to 0 when R tends to infinity.

Then, for any integer $1 \leq i \leq s$, and any function $h^{(s+1)} \in \mathcal{C}\big([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1})\big)$ such that there exists $\lambda \in \mathbb{R}_+$ which satisfies

$$\Big\| h^{(s+1)}(t,Z_{s+1}) \Big\|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))} \leq \lambda \big\| g_{s+1}\big(t,\big|V_{s+1}\big|\big) \big\|_{L^\infty([0,T],L^\infty(\mathcal{D}^\varepsilon_{s+1}))},$$

the function $\mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}$ is well-defined, belongs to $L^\infty\big([0,T]\times\mathcal{D}^\varepsilon_{s+1}\big)$, and we have almost everywhere on $[0,T]$ times$\mathcal{D}^\varepsilon_s$:

$$\Big| \mathcal{C}^\varepsilon_{s,s+1,\pm,i}\mathcal{T}^{s+1,\varepsilon}_t h^{(s+1)}(t,Z_s) \Big| \leq \lambda \varepsilon^{d-1} \frac{\big|\mathbb{S}^{d-1}\big|}{2} \int_{\mathbb{R}^d} \big(|v_i| + |v_{s+1}|\big) g_{s+1}\big(t,\big|V_{s+1}\big|\big)\,\mathrm{d}v_{s+1}.$$

The question of the definition of the spaces $\widetilde{\mathbf{X}}_{\widetilde{\beta},\widetilde{\mu}}$ (see the Definition 2.3 page 16), necessary to solve the hierarchies using a fixed point method, is also discussed in detail in [19], [20]. In particular, there remained unclear points in the literature regarding the stability of such spaces under the action of the collision operators introduced above. In [19], [20] we establish that such spaces cannot be stable if the continuity in time is uniform in s, and we show that by taking (36) as the condition of continuity, we can apply the fixed point, which makes it possible to obtain Theorem 2.4 in the spaces introduced in [19], [20].

3.2 Rigorous derivation of the Boltzmann equation in a domain with boundary: the case of the half-space

A natural generalization of Lanford's theorem consists of studying the case of a domain with boundary. When we prescribe specular reflection as the boundary condition, we then see that the steps of the derivation described in Sections 2.2 and 2.3 can be followed identically, and they provide the same results, up to Duhamel's formulae (42) for hierarchies.

The problem of the control of the recollisions in the case of a domain with boundary presents substantial additional difficulties. In the thesis [19], we considered the simplest case of domain with boundary: the half-space $\mathbb{R}_+ \times \mathbb{R}^{d-1}$, which is therefore delimited by the hyperplane $\{0\} \times \mathbb{R}^{d-1}$. In the case

of the whole Euclidean space, there is only one way for two particles with fixed initial positions to collide, since the particles move in straight lines. In the case of the half-plane, for a pair of particles, each of the two particles may have hit the boundary of the domain, or not hit it, before the collision occurs between the two particles of the pair. We therefore have four configurations to study, which makes the application of the shooting lemmas of [24] more complex and more technical.
The boundary of the domain also produces the following phenomenon, with important consequences, and which are not only of technical nature. As we saw in Section 2.3, we seek to show that only a small quantity of pseudo-trajectories present a recollision. In the case of a domain with boundary, one cannot hope to prove such a result when a particle of the system undergoes a particle adjunction when it is close to the boundary of the domain. We must then, one way or another, get rid of the pseudo-trajectories for which the particles are close to the boundary. However, the respective positions of the particles are never integration variables when we define the integrated transport-collision operators (26), (29), (38), (39).
The method therefore consists of ensuring that the particles do not stay too long near the boundary. To do this, we can carry out a first cut-off in grazing velocities (relative to the boundary of the domain), then a second cut-off in times when the particles are close to the boundary, the sets of times that we obtain are then small thanks to the cut-off in grazing speeds. This method is described in detail in the thesis [19], and summarized in the article [20].
We can then deduce a Lanford's theorem in the case of the half-space, that is, a rigorous derivation theorem of the Boltzmann equation in this domain, with an explicit convergence rate.

Theorem 3.2 (Lanford's theorem in the half-space with specular reflection, D. 2019 [19], [20])**.** *Let* $f_0 : \big\{ x \in \mathbb{R}^d \ / x \cdot e_1 \geq 0 \big\} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ *a continuous probability density such that*

$$f(x,v) \underset{|(x,v)|\rightarrow+\infty}{\longrightarrow} 0 \text{ and } \Big\| f_0(x,v) \exp\Big(\frac{\beta}{2}|v|^2\Big) \Big\|_{L^\infty(\mathbb{R}^{2d})} < +\infty \tag{49}$$

for a certain number $\beta > 0$.
We consider the system of N *hard spheres of diameter* ε *in the half-space (in dimension* d*) with specular reflection, initially distributed according to the density* f_0*, and independent.*
Then, in the Boltzmann-Grad limit $N \rightarrow +\infty$, $N\varepsilon^{d-1} = 1$, *the first marginal* $f_N^{(1)}$ *of the distribution function of the system converges towards the solution* f *of the Boltzmann equation with collision kernel* $b(v,\omega) = (v\cdot\omega)_+$, *with specular reflection as boundary condition, and the initial condition* f_0*, in the following sense:*
for any compact set K *of the phase space* $\{x\cdot e_1 \geq 0\} \times \mathbb{R}^d$ *to a particle such that*

$$K \subset \{x\cdot e_1 > 0\} \times \{v \cdot e_1 \neq 0\}, \tag{50}$$

we have

$$\Big\| \mathbb{1}_K(x,v)\big(f_N^{(1)} - f\big)(x,v) \Big\|_{L^\infty([0,T]\times\{x\cdot e_1>0\}\times\{v\cdot e_1\neq 0\})} \underset{N\rightarrow+\infty}{\longrightarrow} 0. \tag{51}$$

If in addition $\sqrt{f_0}$ *is a Lipschitz function with respect to the position variable, uniformly in the velocity variable, the convergence rate is at most of order* $O(\varepsilon^a)$ *with* $a < 13/128$.

This is the first rigorous derivation result, with an explicit convergence rate, of the Boltzmann equation in a domain with boundary. Let us also observe the locally uniform convergence in the velocity *and* position variables obtained in Theorem 3.2, which is an improvement of the convergence obtained in Theorem 2.5.
The restriction (50) on the compact sets K on which we obtain the uniform convergence (51) is particularly interesting: we see that the boundary of the domain induces a singularity. In the future, we can then consider the following question.

Project 3.1. Is the restriction of the uniform convergence domain more than a technical artifact of the proof? Is it a boundary layer phenomenon, already appearing on the mesoscopic scale, that is to say, when we move from the microscopic description of the gas to the statistical description given by the kinetic equations? What is the true nature of such a restriction?

3.3 Towards domains with a more general geometry: partial results in the case of the disk

With Chiara Saffirio (University of Basel) we have undertaken, since 2020, to extend the result of Lanford's theorem to the case of the disk, assuming once again that the specular reflection takes place at the boundary of the domain. This is the main project of my first post-doctorate, a project on which we are still working. As in the case of the half-space, the different stages of the classical proof adapt without problem and it is possible to obtain the same results as in the case of the domain without boundaries (existence and uniqueness of the solutions of the hierarchies, explicit expression solutions), with the exception of the control of recollisions, which requires in the present a careful study of the trajectories.

The control of recollisions in the case of the disc this time gives rise to notable complications, much more difficult to solve than in the case of the half-space. Indeed, in the case of the disk, as for any domain whose complement is not convex, a particle can a priori bounce a very large number of times against the boundary of the domain before colliding with another particle of the system. Controlling recollisions then amounts to solve a shooting problem, explicitly, in a circular billiard table. More precisely, we had to prove that, for any bouncing number n, any starting point x_1 and any target point x_2 in the disk, only a small set of initial velocities makes it possible to reach the neighborhood $B(x_2, \varepsilon)$ starting from x_1, bouncing exactly n times against the edge of the disk.
We managed to solve this shooting problem, proceeding according to the following steps.

1. Obtaining an upper bound on the number of trajectories from x_1 to x_2 with exactly n bouncings, uniform in x_1 and x_2.
   To obtain such a bound, it is convenient to consider, for a point x_1 that is fixed, all the trajectories starting from this point, and to focus on the lines obtained after exactly n bouncings. One way to consider these lines together is to study the envelope of this family of lines. In the case of a single bouncing, for a starting point x_1 on the edge of the circle, the envelope that we obtain is a curve that is well-known: the cardioid. Figure 4 illustrates this approach. Generally speaking, the envelopes of reflected rays are called *caustic curves*, which can be defined for any starting point x_1, and any number of bouncings n. It turns out that these general caustic curves have already been studied, by Holditch in 1858 [30]. An example of such a curve is shown in Figure 5. We can show that these curves, which we will denote by $\mathcal{H}_n(x_1)$, are algebraic, of degree $\leq Cn$, where C is a universal constant. Therefore, we deduce by a classical theorem of algebraic geometry that the number of tangents to the Holditch curve $\mathcal{H}_n(x_1)$ passing through a given point x_2 is bounded by Cn^2. Now, a tangent to the Holditch curve is by definition one of the lines of the family of which the Holditch curve is the envelope. In other words, by definition, for any trajectory starting from x_1, the line which contains the part of the trajectory between its n-th and its $(n+1)$-th bouncings is a tangent to the Holditch curve $\mathcal{H}_n(x_1)$. As a consequence, we deduce the uniform bound, in x_1 and x_2, on the number of trajectories which reach x_2 from x_1 after exactly n bouncings.

2. Show that any trajectory from x_1 to $\widetilde{x}_2 \in B(x_2, \varepsilon)$ is obtained by slightly perturbing the direction of the initial velocity of a trajectory going *exactly* from x_1 to x_2.
   The objective of this point is clear: we want to characterize the set of trajectories going from x_1 to any point of $B(x_2, \varepsilon)$ in exactly n bouncings. To do this, we consider a given trajectory,

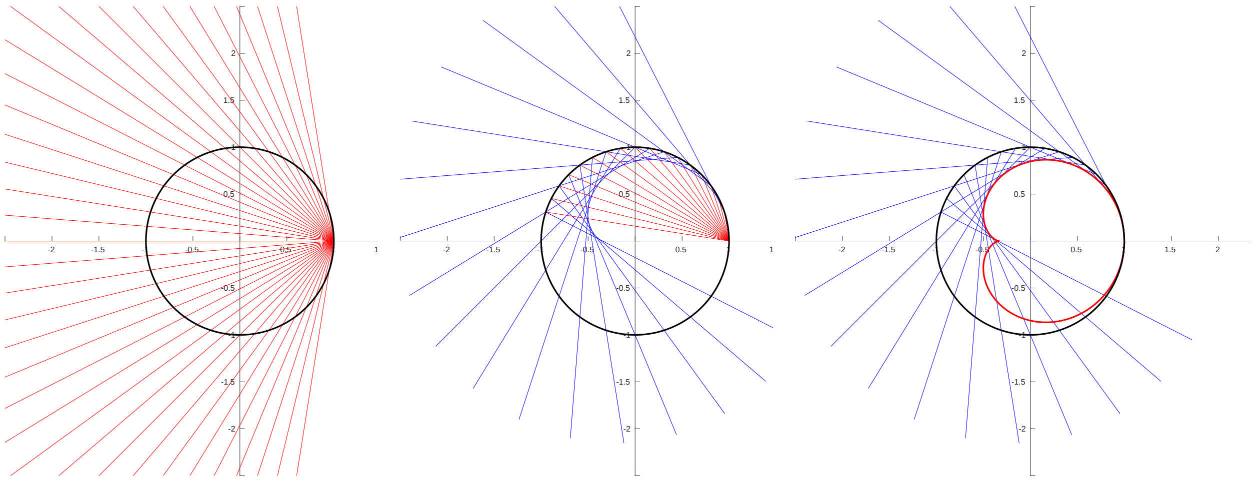


Figure 4: Representation of the first part of the trajectories starting from a given point lying on the circle on the left, in the center the trajectories obtained after a first reflection, and on the right the envelope of these reflected lines: the cardioid.

which goes from x_1 to x_2 in n bouncings. We want to show that only a small perturbation of the direction of the initial velocity of this trajectory is necessary to reach any point of $B(x_2, \varepsilon)$ in n bouncings. More precisely, we would like to show that to cover a small disk of size ε, a perturbation of order ε only is sufficient to cover the targeted disk. This is an intuitive result. However, it is not true in general. Indeed, on the line which contains the n-th part of the trajectory from x_1 to x_2 there is a particular point, which we will denote by z_n, and which is the intersection of this line with the Holditch curve. In other words, z_n is the point of tangency of the trajectory with the Holditch curve $\mathcal{H}_n(x_0)$. Now, a caustic curve can be seen, intuitively, as the locus of the points of intersection of curves $\mathcal{C}_\lambda$, $\mathcal{C}_{\lambda'}$ (from the family of curves $(\mathcal{C}_\lambda)_\lambda$) which are infinitely close (that is, with $\lambda \neq \lambda'$, but $|\lambda' - \lambda|$ very small). Thus, close to z_n, a perturbation of the initial direction of order ε is not enough to cover a targeted disk of radius ε.
On the other hand, we can show that for any $\delta > 0$, provided that the point x_2 is at a sufficient distance from the point z_n on the Holditch curve, then indeed a perturbation of the direction of the initial velocity of order $\varepsilon^{1-\delta}$ is enough to completely cover the targeted disk $B(x_2, \varepsilon)$.
Consequently, any trajectory of x_1 which reaches in n bouncings any point $\widetilde{x}_2 \in B(x_2, \varepsilon)$ can be perturbed in order to obtain a trajectory reaching exactly x_2 in n bouncings, and so we deduce the characterization of the initial velocities of the trajectories from x_1 to $B(x_2, \varepsilon)$ in n bouncings.

3. The last step consists of ensuring that (up to proceed to a cut-off of a small set of pseudo-trajectories) we can actually assume that the center x_2 of the target disk is generally quite far from the point z_n, described in the previous step. This third step is obtained by using the transversality of the trajectories with the Holditch curve.

Our proof allows to control recollisions in the case of the disk, in a quantitative manner. However, our proof currently only applies only to the pre-collisional case. Our next objective is now to adapt this proof to the post-collisional case, which we hope to achieve within a few weeks.

Project 3.2. Extend the above proof to the post-collision case, which will allow us to complete the proof of the rigorous derivation of the Boltzmann equation in the disk.

These partial results gave rise to a presentation at a conference in Oberwolfach last September (workshop "Classical and Quantum Mechanical Models of Many- Particle Systems", presentation which appears in the Oberwolfach reports [42]), as well as the presentation of a poster at the PSPDE XI conference in Lisbon.

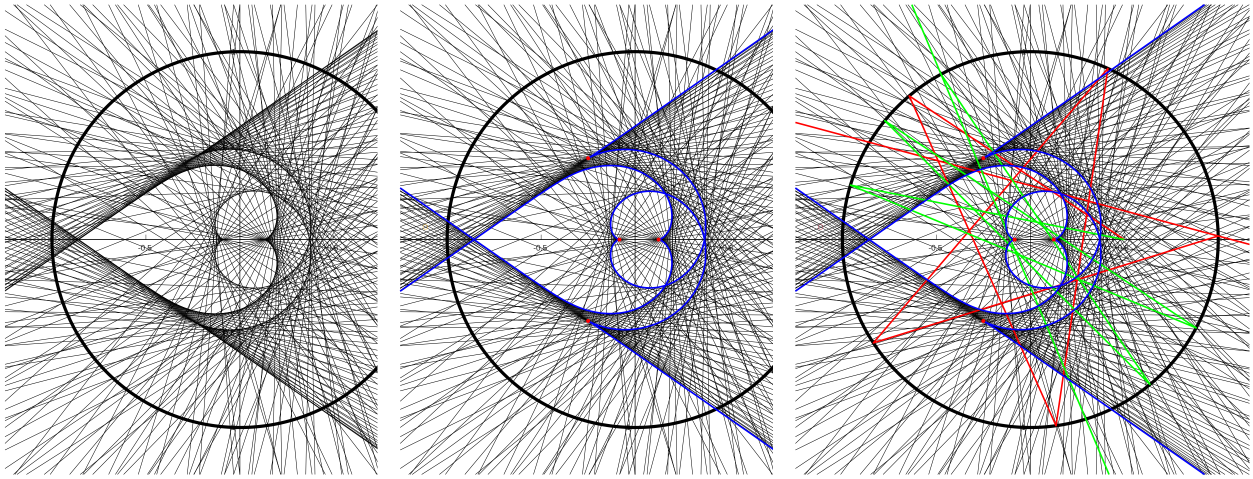


Figure 5: Representation of the trajectories starting from the point $x_1 = (0.5, 0)$ obtained after 5 reflections on the left, in the center the associated Holditch curve $\mathcal{H}_5(x_1)$ is plotted, and on the right we added some particular trajectories, from their common starting point, up to their fifth bouncing.

4 Contributions to the theory of inelastic particles

The results obtained in this section are the results of my second post-doctoral fellowship, under the supervision of Juan Velázquez (University of Bonn). Our initial motivation was to establish an Alexander theorem for systems of inelastic particles, which collide according to the law (19) (see Section 1.4, and in particular its last paragraph, for a presentation of the context of this important problem in the case of granular gases). This problem has proven to be very difficult, even for systems consisting of small numbers of particles.

4.1 Study of the inelastic collapse of 3 particles in dimension $d \geq 2$: classification of singularities and explicit construction of collapses

In order to better understand the phenomenon of inelastic collapse, we studied the case of a system of three particles, in dimension $d \geq 2$. Let us name these particles 0, 1, 2. In [56], Zhou and Kadanoff studied this system, obtaining two necessary conditions on the final particle geometry as a function of the restitution coefficient r. In particular, if we call θ the angle $(x_1 - x_0, x_2 - x_0)$ (that is, if we assume that the particle 0 is in contact with each of the two other particles 1 and 2 at the moment of collapse, θ is the angle formed by the particles 1 and 2 around of the central particle 0), these authors showed that there cannot exist a stable collapse (with respect to perturbation of the initial data) if the angle θ does not verify the inequality:

$$-\cos\theta > \frac{2r^{1/3}(1+r^{1/3})}{1+r}. \tag{52}$$

Our goal was to establish a solid mathematical theory of such a system.

First, we showed that when a system of three particles undergoes an inelastic collapse, the pairs of particles which collide an infinite number of times are in contact at the limit. We have also shown the convergence towards 0 of the normal components of the relative velocities. Finally, we determined the asymptotic behaviors of the different variables used to describe the particle system, in the collapse limit. We found in particular, in a rigorous manner, the fact that all the variables which tend towards 0 are estimated by the normal components of the relative velocities, and by the differences between

the times of consecutive collisions.
Let us emphasize that these general results are obtained without assuming anything but the fact that a collapse indeed takes place (i.e., assuming only that an infinite number of collisions occur in a finite time, without assume nothing else). This is an important difference from [56], where the authors assumed a priori that the order of collisions is given by the infinite repetition of the pair of consecutive collisions 0-1, 0-2.
Let us observe that the order of collisions of a system of three particles is necessarily, in dimension 1, the order considered by Zhou and Kadanoff. On the other hand, in higher dimensions, a priori any order of collisions is possible. From the general results mentioned above, we in fact deduced a complete classification of the possible singularities that constitutes a collapse. More precisely, we proved that if a collapse occurs, it can only occur according to one of the only two possible orders of collisions. If an infinite number of collisions occur, the order of collisions becomes, after a certain (finite) number of collisions, a periodic order. The first order corresponds to the case studied by Zhou and Kadanoff, and corresponds to a perturbation of the case one-dimensional case. The second possible order is the repetition of the period (up to relabel the particles) 0-1, 0-2, 1-2. Note that such a collision order can *never* be associated with a one-dimensional system. In this sense, it is the description of a typically d-dimensional collapse phenomenon, with $d \geq 2$. To our knowledge, no such phenomenon has been previously described in the literature. In particular, if such a collapse occurs, the three particles are all in contact at the limiting time of collapse, forming an equilateral triangle. There is no other possible collision order.
Note also that knowing the order of collisions allows us to study the final state of the system. In particular, we can deduce constraints on the relative velocities at the moment of collapse. In the case of the triangular collapse, we obtained that the relative velocities are extremely constrained, which strongly suggests that if such a triangular collapse exists, it must be unstable with respect to perturbations of the initial configuration. However, we did not constructed explicitly an initial configuration that produces a triangular collapse. A reasonable objective, undoubtedly relatively calculative but elementary, is the following.

Project 4.1. Determine whether the triangular collapse is indeed occurring. If yes, construct an initial configuration of the particles such that the dynamics of the system starting from this configuration produces a triangular collapse, and if possible, describe the rate of convergence of the different variables describing the dynamical system.

Finally, as an application of the results on the asymptotic behavior of the different variables in the neighbourhood of the collapse, we explicitly constructed initial configurations which lead to the inelastic collapse, following the order of collisions 0-1, 0-2. We showed that such a collapse is stable, in the sense that perturbations of the configurations that we provided still lead to a collapse. This is the first explicit construction of stable configurations leading to a collapse in the literature, to our knowledge. The fact that these configurations are stable explains why collapse has been observed so often during the numerous numerical simulations found in the literature (see for example [39], [43]).
To conclude, let us note that we managed to construct collapses such that the final geometry of the system is prescribed a priori, with arbitrary precision, within the framework of the conditions already stated by Zhou and Kadanoff in [56] in the following sense. For any angle $\theta_0 \in [\pi/2, \pi]$ which satisfies the condition (52) of Zhou and Kadanoff, we produced initial configurations such that the final angle at the moment of collapse is arbitrarily close to θ_0. In particular, for r small enough, we produced configurations whose final angle is arbitrarily close to $\pi/2$, which constitutes an extreme deformation of the one-dimensional case.
These results are presented in the recent pre-print [21].

In another work [23], we studied in more detail the collapse of three particles, assuming that the order of collisions is the repetition of the period 0-1, 0-2. The objective remains the same: to prove an Alexander theorem for such a system of particles. We therefore sought to determine which initial configurations led to a collapse so that the tangential components of the relative velocities at the moment of the collapse were not zero. In such a case, it would be possible to continue the dynamics of the particles beyond the time limit of the collapse: the agglomeration of particles would then "dissipate" by itself.

The approach adopted in this work consists of using the asymptotic behaviors of the variables to simplify the dynamical system which fully describes the evolution of particles. This complete dynamical system is too complicated to be studied directly, since for particles of $\mathbb{R}^2$, we have a system of dimension 7, of dimension 11 in the case of $\mathbb{R}^3$.

However, the study of the asymptotic behavior of the variables makes it possible to reduce the complete dynamical system, by studying the evolution of the leading order terms. We then obtained a reduced system of dimension 2, the evolution of which leads the rest of the other variables of the complete dynamical system.

The study of the reduced system is not simple, and it is necessary to identify particular regimes in which the dynamics can be simplified even further. Inspired by the article by Zhou and Kadanoff, we identify a regime in which the dynamical system ultimately reduces to a system of dimension 1, which we can therefore study completely. We then attempted to characterize, in the two-dimensional phase space of the reduced dynamical system, the orbits for which the Zhou-Kadanoff regime is valid in the limit. More precisely, we obtained the following results.

- We proved that the Zhou-Kadanoff regime is attracting in the phase space of the reduced dynamical system, at least in a non-trivial region.

- Based on the numerical simulation of the orbits of the reduced dynamical system, we conjectured that the Zhou-Kadanoff regime is the only possible stable regime, that is to say, for any part of the phase space of positive measure, either the orbits are not globally defined, or they asymptotically verify the Zhou-Kadanoff regime. In addition, we identified a curve in the phase space, which is a separatrix: this curve separates the phase space into two parts, on one side the orbits which are not globally defined, on the other the orbits which verify the Zhou-Kadanoff regime. We hope to be able to prove these conjectures regarding the reduced dynamical system in the near future.

- Finally, we identified a formal limit which allows us to rewrite the equations of the reduced dynamical system. For this new system obtained in this limit, which we called *low energy limit*, we proved the conjectures of the previous point. In particular, we show that there does exist a separatrix, which clearly separates the phase space between the orbits which satisfy the Zhou-Kadanoff regime on the one hand, and on the other hand the orbits which become singular in finite time.

The next questions to be addressed are as follows. The first project seems reasonable, since it largely follows the ideas developed in the generic case. The second project aims to obtain a technical result, but which in fact justifies the reduction to a system of dimension 2. Finally, the third project, undoubtedly difficult, will perhaps require new tools (computer-assisted proofs, to determine the separatrix?)

Project 4.2. Conduct the same study in the case when one of the two relative velocities tends towards zero at the time of collapse, then in the case where both relative velocities tend towards zero: reduce the size of the system, identify the generic regime, and show its stability, at least in a non-trivial part of the phase space.

Project 4.3. Rigorously justify that, in the collapse regime, the orbits of the complete dynamical system are correctly approximated by the orbits of the reduced system.

Project 4.4. Prove the characterization of the orbits of the reduced system which verify the Zhou-Kadanoff regime. In particular, show that there indeed exists a separating curve, probably composed of heteroclinic trajectories, which separates the phase space between the asymptotically Zhou-Kadanoff orbits, and those which become singular in finite time.

4.2 An example of an inelastic particle system whose flow preserves the measure in the phase space, but whose kinetic energy is not conserved

In the recent pre-print [22], we constructed an example of a particle system with two properties, a priori incompatible. On the one hand, the collisions between particles are inelastic, and therefore the system dissipates kinetic energy. On the other hand, the flow associated with the particle dynamics preserves the measure in the phase space. This property is surprising, since in the case of elastic hard spheres, the measure is preserved in the phase space (this is an important property used to establish Alexander's theorem [1], which ensures the well-posed character for almost any trajectory). In the inelastic case (19), the measure is not preserved, which prevents us from reproducing the proof of Alexander's theorem in this case.
The model that we built in [22] is a two-dimensional model, and we assumed that a fixed quantity $\varepsilon_0 > 0$ of kinetic energy is dissipated during each collision, according to the law:

$$\begin{cases} v_i' &= \frac{v_i+v_j}{2} - \sigma\sqrt{\frac{|v_j-v_i|^2}{4} - \varepsilon_0}, \\ v_j' &= \frac{v_i+v_j}{2} + \sigma\sqrt{\frac{|v_j-v_i|^2}{4} - \varepsilon_0}, \end{cases} \tag{53}$$

with

$$\sigma = \frac{v_j - v_i}{|v_j - v_i|} - 2\left(\frac{(v_j - v_i)}{|v_j - v_i|}\cdot\omega\right)\omega. \tag{54}$$

Such a law can be interpreted as follows: we model excited particles, which when they collide emit a photon of energy. For this reason, we called this model the *system of hard spheres with emission*.
It is important to note that the system preserves the measure in the phase space only in dimension 2. In [22], we give an interpretation of this property of measure conservation in terms of conformal mappings, which allows us to define other models of inelastic particles, in any dimension, whose flow preserves the measure in the phase space.

An important application of the properties of the system of inelastic hard spheres with emission is the following result.

Theorem 4.1 (Alexander's theorem for inelastic hard spheres with emission, D., Velázquez 2024 [22])**.** *Let N be a strictly positive integer.*
Then, the dynamics of the system of N inelastic hard spheres with emission is globally defined for almost any initial configuration in dimension $d = 2$. In other words, for almost every initial configuration (with respect to the Lebesgue measure) $Z_N = (x_1, v_1, \dots, x_N, v_N)$ of N particles, the evolution of the system from this initial configuration Z_N is well-defined for all times $t \geq 0$, only involves inertial movements or binary collisions, and moreover for all $T > 0$, the system does not undergo more than a finite number of collisions over the time interval $[0, T]$.

To the best of our knowledge, this is the first example of an Alexander's theorem for an inelastic particle model. This result prompts us to consider the following question.

Project 4.5. Prove an Alexander theorem for inelastic particles whose restitution coefficient is variable, and tends towards 1 for small relative speeds (viscoelastic case, considered as more realistic in the physics literature).

5 Ongoing projects, and perspectives

5.1 Ongoing projects and collaborations

In parallel with the project started with Chiara Saffirio, already mentioned in Section 3.3 and which is a long-term collaboration, I am involved in the following ongoing projects.

Project 5.1. Study the derivation of the Boltzmann equation in a domain, which is the complement of a convex obstacle. I am leading this project alone, in a relatively advanced state and on which I started working during my first post-doctoral fellowship in Basel.

Project 5.2. Study of the coagulation equation, in the case of the additive kernel $K(x,y) = x+y$, with a source term. This project, started during my second post-doctoral fellowship in Bonn with Eugenia Franco (University of Bonn), aims to study the long-time behavior of the Smoluchowski coagulation equation. In particular, we already know that in the case of the constant kernel $K(x,y) = 1$, a periodic source in time produces a periodic solution in time [47]. But on the other hand, the long-time behavior of the coagulation equation, without source, for constant, additive or multiplicative kernel has been studied in detail, in particular in [40]. It is remarkable that the solutions converge towards a self-similar profile. In our case, the question is to determine whether the periodicity in time of the source, is also a property that holds, or not, for the solution of the equation for large time. In other words, we seek to know which effect dominates between the attraction towards the self-similar profile, and the forcing of oscillations due to the source.

Project 5.3. Study of the derivation of the Boltzmann equation from discrete velocity particle models (Uchiyama model). This project was started with Nathalie Ayi (LJLL) in November 2023. It is known that the Uchiyama particle model, in the Boltzmann-Grad limit, does not converge towards the Broadwell equation (the equivalent of the Boltzmann equation, for particles with discrete velocities), see for example [50], [51]. But certain authors ([11], [18]) have managed to derive the Broadwell equation from particle models, discrete in position and speed (HPP model [29]), provided to introduce stochasticity into the dynamics of the particles. We aim to understand if such an approach can provide similar results starting from the Uchiyama model (which describes particles with discrete velocities, but continuous positions).

Project 5.4. Study of the system of four inelastic particles in dimension 1. This work is a continuation of the master thesis [31] of Eleni Hübner-Rosenau (now at the University of Regensburg), which I supervised and which was defended in November 2023. With Eleni Hübner-Rosenau, based on the article [16], we seek to determine what are the possible orders of collisions for such particle systems. We have already collected some partial results, and we have already carried out a large number of numerical simulations of such particle systems.

5.2 Insertion of my research themes into the University

My field of research belongs to the field of the analysis of PDEs, in particular kinetic equations, and also belongs to the field of dynamical systems, in particular, the field of interacting particle systems. Italy has one of the strongest communities of experts in kinetic equations, which constitutes for me an extremely stimulating environment. Besides, the study of the derivation of the Boltzmann equation maintains strong links with the equations of fluid mechanics when we study hydrodynamic limits, in particular with the Euler and Navier-Stokes equations. Furthermore, it would be interesting, within the framework of the project 3.1, to explore the connections between Lanford's proof and Prandtl's system, which describes the interactions between a fluid and a solid structure. In the same way, fluid mechanics equations obtained in the context of granular materials constitute a natural area of interaction with the research I have carried out so far.
Concerning particle systems, especially in the inelastic case, it should be noted that many phenomena

have been observed numerically, and within the framework of the project 5.4 we have made an extensive use of simulations. This subject therefore invites interaction with experts of numerics (especially, about discrete dynamical systems), and important questions at the interface with this field arise: for example, what can we say about the numerical stability of the simulations? Is there a shadowing lemma in the inelastic case?
Of course, we can also approach particle systems with the classical tools of dynamical systems (invariant measures, ergodicity, mixing property, existence of attractors, etc.). I do not believe that these questions have been explored so far in the inelastic case, and the results obtained in the elastic case do not allow to claim a deep understanding of the phenomena at stake (what about the correlation between particles, so important to understand the Boltzmann equation?). I would also be delighted to initiate collaborations in these directions.
We can also consider the framework of particle systems whose dynamics are stochastic (project 5.3). These questions naturally lead to discussions with specialists of probability.
Finally, the project 5.2 started with Eugenia Franco concerns the coagulation equation, which belongs to the world of biomathematics.

6 Comments on past teaching experiences and perspectives

6.1 Past teaching experiences

Since the beginning of my thesis in 2016, I have had to teach to diverse groups, ranging from undergraduate students to fellow researchers. I have always taken great pleasure in passing on knowledge.

During my three years of thesis I was in charge of exercise classes at the University of Paris Diderot. During my first year, I led exercise corrections for classes of biologists, then chemists, in both cases in their first year of study. Here, the exercise consisted of addressing an audience that was not comfortable with the subject, most of the students in these groups not seeing Mathematics as an interesting subject for itself, but only as a tool. With such an audience one must take this into account, and one has often to move forward in a minefield, when certain students have had a negative experience with the subject. But with the time available during weekly tutorial sessions, I believe it is possible not only to respect the program, but also to go over the basics when necessary. The first year program is basic enough to start from scratch, and the examples available are numerous enough to be able to illustrate each concept.
The last two years of my thesis, I was then in charge of tutorials for undergraduate students in fundamental or applied mathematics in their last year, for the introductory course on ordinary differential equations lectured Davide Barilari. The material is very rich, since we have covered the systematic (almost mechanical) approach to the resolution of linear equations in dimension 1, with constant coefficients in the case of order 2, then the linear systems in any dimension on the one hand, and on the other hand the consequences of the Cauchy-Lipschitz theorem (subject which exposes more the students to the qualitative properties of solutions).
In parallel with this teaching, I was also in charge of the "in-depth" teaching unit, which was aimed at first-year Mathematics students. On a voluntary basis, we solved advanced exercises every week, and I was free to choose the themes of the sessions. The first year, for example, I organized the course around a guiding theme: finding Newton's result about the elliptical trajectories of the planets. I found the subject appropriate, since it was necessary to practice, in turn, differential equations (and therefore, differential and integral calculus), conics (and therefore, geometry and algebra), and even questions of stability of dynamical systems (for example, we looked at Lagrange points, which require to use Taylor expansions). None of these subjects were outside the scope of the program, and they all required a good mastery of the objects that the students had just discovered.

Subsequently, I continued to teach, in Basel then in Bonn. At Paris Diderot all the courses were in French, I subsequently learned to work in English. During the first semester of 2019-2020, I also corrected German papers, handed in every week. In Basel, I was an assistant for Marcus Grote's introductory numerical analysis courses, Enno Lenzman's analysis courses (compactness, completeness, connectedness, metric spaces, topology, and differential calculus), for the lecture on measure theory and integration of Chiara Saffirio, and finally an introduction to PDEs and Sobolev spaces by Gianluca Crippa. I was also a lecturer, for a master's unit, during half a semester, on kinetic equations: Chiara Saffirio covered the part on transport and Vlasov equations, and I presented the Boltzmann equation. This experience was stimulating and demanding: I had to address master's students who were not all familiar with functional analysis. I therefore built the course and illustrated the main phenomena around the Boltzmann equation using, among others, the article by Carleman [10], quite elementary, but in which we find the ideas applied later in the framework of modern functional analysis (a priori estimates, construction of solutions by schemes of sub- and super-solutions which converge and which are based on the positivity of the operator, etc.). Preparing the course and weekly exercises based only on scientific literature (the books on the subject were mostly based on tools too advanced for the students to be used) taught me a lot, not only as a teaching experience, but I also took the opportunity to explore in more detail many questions related to the subject of the lecture.
Finally, in Bonn, where my contract did not present mandatory teaching duties, I continued to discover new forms of teaching. At this University, professors and post-docs are free to organize courses on the theme of their choice. In addition to the introductory master's course on the Boltzmann equation, which I gave during the fall semester 2023 and which I enriched compared to the one given in Basel, I was an assistant for several courses (dynamical systems, asymptotic methods for differential equations and dynamical systems) and organized a "Graduate Seminar": around the theme of the hydrodynamic limits of the Boltzmann equation, each week a student was asked to present a recent article of the domain.
In Basel as in Bonn, students evaluate course and exercise session leaders via anonymous questionnaires. I have always received very positive feedback: in Bonn for example, during the course on the Boltzmann equation, 6 out of 6 students answered the question "Halten Sie den*die Dozent*in für lehrpreiswürdig?" (do you think the lecturer deserves a teaching award?) by "Ja, definitiv".
Generally speaking, I remember from my stays abroad the discovery of different ways of teaching. From each of these approaches, I believe that we can derive different and complementary advantages. In any case, I am happy to have seen how our neighbors work, and I hope to have improved through contact with them.

Finally, I would like to mention my experiences supervising master's theses: I was lucky to have this responsibility in Bonn. I supervised Eleni Hübner-Rosenau's master's thesis, defended in November 2023 [31], on a subject that I chose in relation to the research project that I am leading in Bonn: problems around the dynamics of inelastic particles in dimension 1. Eleni Hübner-Rosenau is now continuing her studies, preparing a thesis at the University of Regensburg. With Eugenia Franco (University of Bonn), since autumn 2023, I have been co-supervising Daniel Happ's master's thesis, on growth-fragmentation equations. I find this experience particularly enriching.

6.2 Perspectives about teaching

In the future, I would like to continue my teaching activities. I consider that this activity ideally complement the work of a researcher.
I would be happy to teach, both in bachelor's and master's degrees. I will be able to do so in English without difficulty, and I believe that I could be able to do it also in Italian in a reasonable amount of time (for a French native speaker, I think that the exercise is probably not so hard).
In addition, I would happily take part in the activities of supervising student dissertations, which I

have enjoyed so much so far. I am thinking, in the short term, of bachelor's and master's theses, but also in the longer term, of more important responsibilities, such as the supervision of theses.

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